MATRIX EXTENSION WITH SYMMETRY AND ITS APPLICATION TO FILTER BANKS*

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Abstract. Let P be an $r \times s$ matrix of Laurent polynomials with symmetry such that $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible. The matrix extension problem with symmetry is to find an $s \times s$ square matrix P_e of Laurent polynomials with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ (that is, the submatrix of the first r rows of P_e is the given matrix P), P_e is paraunitary satisfying $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$, and the symmetry of P_e is compatible. Moreover, it is highly desirable in many applications that the support of the coefficient sequence of P_e can be controlled by that of P. In this paper, we completely solve the matrix extension problem with symmetry and provide a step-by-step algorithm to construct such a desired matrix P_e from a given matrix P. Furthermore, using a cascade structure, we obtain a complete representation of any $r \times s$ paraunitary matrix P having compatible symmetry, which in turn leads to an algorithm for deriving a desired matrix P_e from a given matrix P. Matrix extension plays an important role in many areas such as electronic engineering, system sciences, applied mathematics, and pure mathematics. As an application of our general results on matrix extension with symmetry, we obtain a satisfactory algorithm for constructing symmetric paraunitary filter banks and symmetric orthonormal multiwavelets by deriving high-pass filters with symmetry from any given low-pass filters with symmetry. Several examples are provided to illustrate the proposed algorithms and results in this paper.

Key words. Matrix extension, symmetry, Laurent polynomials, paraunitary filter banks, orthonormal multiwavelets.

AMS subject classifications. 15A83, 15A54, 42C40, 15A23

1. Introduction and Main Results. The matrix extension problem plays a fundamental role in many areas such as electronic engineering, system sciences, mathematics, and etc. To mention only a few references here on this topic, see [1, 2, 3, 5, 6, 7, 9, 12, 14, 15, 16, 17, 19, 20]. For example, matrix extension is an indispensable tool in the design of filter banks in electronic engineering ([13, 14, 19, 20]) and in the construction of multiwavelets in wavelet analysis ([1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 15, 18, 16]). In order to state the matrix extension problem and our main results on this topic, let us introduce some notation and definitions first.

Let $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k, z \in \mathbb{C}\backslash\{0\}$ be a Laurent polynomial with complex coefficients $p_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. We say that p has *symmetry* if its coefficient sequence $\{p_k\}_{k\in\mathbb{Z}}$ has symmetry; more precisely, there exist $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \varepsilon p_k, \qquad \forall\, k \in \mathbb{Z}. \tag{1.1}$$

If $\varepsilon = 1$, then p is symmetric about the point $c/2$; if $\varepsilon = -1$, then p is antisymmetric about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator $\mathcal{S}$ defined by

$$\mathcal{S}\mathsf{p}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(1/z)}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{1.2}$$

When p is not identically zero, it is evident that (1.1) holds if and only if $\mathcal{S}\mathsf{p}(z) = \varepsilon z^c$. For the zero polynomial, it is very natural that $\mathcal{S}0$ can be assigned any symmetry

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pattern; that is, for every occurrence of $\mathcal{S}0$ appearing in an identity in this paper, $\mathcal{S}0$ is understood to take an appropriate choice of εz^c for some $\varepsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$ so that the identity holds. If P is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator $\mathcal{S}$ to each entry of P, that is, $\mathcal{S}\mathsf{P}$ is an $r \times s$ matrix such that $[\mathcal{S}\mathsf{P}]_{j,k} := \mathcal{S}([\mathsf{P}]_{j,k})$, where $[\mathsf{P}]_{j,k}$ denotes the (j,k)-entry of the matrix P throughout the paper.

For two matrices P and Q of Laurent polynomials with symmetry, even though all the entries in P and Q have symmetry, their sum $\mathsf{P} + \mathsf{Q}$, difference $\mathsf{P} - \mathsf{Q}$, or product PQ, if well defined, generally may not have symmetry any more. This is one of the difficulties for matrix extension with symmetry. In order for $\mathsf{P} \pm \mathsf{Q}$ or PQ to possess some symmetry, the symmetry patterns of P and Q should be compatible. For example, if $\mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$, that is, both P and Q have the same symmetry pattern, then indeed $\mathsf{P} \pm \mathsf{Q}$ has symmetry and $\mathcal{S}(\mathsf{P} \pm \mathsf{Q}) = \mathcal{S}\mathsf{P} = \mathcal{S}\mathsf{Q}$. In the following, we discuss the compatibility of symmetry patterns of matrices of Laurent polynomials. For an $r \times s$ matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$, throughout the paper we denote

$$\mathsf{P}^*(z) := \sum_{k\in\mathbb{Z}} P_k^* z^{-k} \quad \text{with} \quad P_k^* := \overline{P_k}^T, \qquad k \in \mathbb{Z}, \tag{1.3}$$

where $\overline{P_k}^T$ denotes the transpose of the complex conjugate of the constant matrix P_k in $\mathbb{C}$. We say that *the symmetry of* P *is compatible* or P *has compatible symmetry*, if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta_2(z), \tag{1.4}$$

for some $1 \times r$ and $1 \times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry. For an $r \times s$ matrix P and an $s \times t$ matrix Q of Laurent polynomials, we say that (P, Q) *has mutually compatible symmetry* if

$$\mathcal{S}\mathsf{P}(z) = (\mathcal{S}\theta_1)^*(z)\mathcal{S}\theta(z) \quad \text{and} \quad \mathcal{S}\mathsf{Q}(z) = (\mathcal{S}\theta)^*(z)\mathcal{S}\theta_2(z) \tag{1.5}$$

for some $1 \times r$, $1 \times s$, $1 \times t$ row vectors $\theta_1, \theta, \theta_2$ of Laurent polynomials with symmetry. If (P, Q) has mutually compatible symmetry as in (1.5), then it is easy to verify that their product PQ has compatible symmetry and in fact $\mathcal{S}(\mathsf{PQ}) = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$.

For a matrix of Laurent polynomials, another important property is the support of its coefficient sequence. For $\mathsf{P} = \sum_{k\in\mathbb{Z}} P_k z^k$ such that $P_k = \mathbf{0}$ for all $k \in \mathbb{Z}\backslash[m, n]$ with $P_m \neq \mathbf{0}$ and $P_n \neq \mathbf{0}$, we define its coefficient support to be $\mathrm{coeffsupp}(\mathsf{P}) := [m, n]$ and the length of its coefficient support to be $|\mathrm{coeffsupp}(\mathsf{P})| := n - m$. In particular, we define $\mathrm{coeffsupp}(\mathbf{0}) := \emptyset$, the empty set, and $|\mathrm{coeffsupp}(\mathbf{0})| := -\infty$. Also, we use $\mathrm{coeff}(\mathsf{P}, k) := P_k$ to denote the coefficient matrix (vector) P_k of z^k in P. In this paper, $\mathbf{0}$ always denotes a general zero matrix whose size can be determined in the context.

The Laurent polynomials that we shall consider in this paper have their coefficients in a subfield $\mathbb{F}$ of the complex field $\mathbb{C}$. Let $\mathbb{F}$ denote a subfield of $\mathbb{C}$ such that $\mathbb{F}$ is closed under the operations of complex conjugate of $\mathbb{F}$ and square roots of positive numbers in $\mathbb{F}$. In other words, the subfield $\mathbb{F}$ of $\mathbb{C}$ satisfies the following properties:

$$\bar{x} \in \mathbb{F} \quad \text{and} \quad \sqrt{y} \in \mathbb{F}, \qquad \forall\, x, y \in \mathbb{F} \quad \text{with} \quad y > 0. \tag{1.6}$$

Two particular examples of such subfields $\mathbb{F}$ are $\mathbb{F} = \mathbb{R}$ (the field of real numbers) and $\mathbb{F} = \mathbb{C}$ (the field of complex numbers).

Now, we introduce the general matrix extension problem with symmetry. Throughout the paper, r and s denote two positive integers such that $1 \leqslant r \leqslant s$. Let P be an

$r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible, where I_r denotes the $r \times r$ identity matrix. The matrix extension problem with symmetry is to find an $s \times s$ square matrix P_e of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$ (that is, the submatrix of the first r rows of P_e is the given matrix P), the symmetry of P_e is compatible, and $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, P_e is paraunitary). Moreover, in many applications, it is often highly desirable that the coefficient support of P_e can be controlled by that of P in some way.

In this paper, we study this general matrix extension problem with symmetry and we completely solve this problem as follows:

THEOREM 1. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that* (1.6) *holds. Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that the symmetry of P is compatible and $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$. Then there exists an $s \times s$ square matrix P_e, which can be constructed by Algorithm 1 in section 2 from the given matrix P, of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(i) *$[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$, that is, the submatrix of the first r rows of P_e is P;*
(ii) *P_e is paraunitary: $\mathsf{P}_e(z)\mathsf{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$;*
(iii) *The symmetry of P_e is compatible;*
(iv) *The coefficient support of P_e is controlled by that of P in the following sense:*

$$|\text{coeffsupp}([\mathsf{P}_e]_{j,k})| \leqslant \max_{1 \leqslant n \leqslant r} |\text{coeffsupp}([\mathsf{P}]_{n,k})|, \qquad 1 \leqslant j, k \leqslant s. \tag{1.7}$$

Theorem 1 on matrix extension with symmetry is built on a stronger result which represents any given paraunitary matrix having compatible symmetry by a simple cascade structure. The following result leads to a proof of Theorem 1 and completely characterizes any paraunitary matrix P in Theorem 1.

THEOREM 2. *Let P be an $r \times s$ matrix of Laurent polynomials with coefficients in a subfield $\mathbb{F}$ of $\mathbb{C}$ such that* (1.6) *holds. Then $\mathsf{P}(z)\mathsf{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of P is compatible as in* (1.4)*, if and only if, there exist $s \times s$ matrices $\mathsf{P}_0, \ldots, \mathsf{P}_{J+1}$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(1) *P can be represented as a product of $\mathsf{P}_0, \ldots, \mathsf{P}_{J+1}$:*

$$\mathsf{P}(z) = [I_r, \mathbf{0}]\mathsf{P}_{J+1}(z)\mathsf{P}_J(z) \cdots \mathsf{P}_1(z)\mathsf{P}_0(z); \tag{1.8}$$

(2) *$\mathsf{P}_j, 1 \leqslant j \leqslant J$ are elementary: $\mathsf{P}_j(z)\mathsf{P}_j^*(z) = I_s$ and* $\text{coeffsupp}(\mathsf{P}_j) \subseteq [-1, 1]$*;*
(3) *$(\mathsf{P}_{j+1}, \mathsf{P}_j)$ has mutually compatible symmetry for all $0 \leqslant j \leqslant J$;*
(4) *$\mathsf{P}_0 = \mathsf{U}_{\mathcal{S}\theta_2}^*$ and $\mathsf{P}_{J+1} = diag(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$, where $\mathsf{U}_{\mathcal{S}\theta_1}$, $\mathsf{U}_{\mathcal{S}\theta_2}$ are products of a permutation matrix with a diagonal matrix of monomials, as defined in* (2.2)*;*
(5) *$J \leqslant \max_{1 \leqslant m \leqslant r, 1 \leqslant n \leqslant s} \lceil |\text{coeffsupp}([\mathsf{P}]_{m,n})|/2 \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.*

The representation in (1.8) (without symmetry) is often called the cascade structure in the literature of engineering, see [13, 14, 19]. In the context of wavelet analysis, matrix extension without symmetry has been discussed by Lawton, Lee and Shen in their interesting paper [15] and a simple algorithm has been proposed there to derive a desired matrix P_e from a given row vector P of Laurent polynomials without symmetry. In electronic engineering, an algorithm using the cascade structure for matrix extension without symmetry has been given in [19] for filter banks with perfect reconstruction property. The algorithms in [15, 19] mainly deal with the special case that P is a row vector (that is, $r = 1$ in our case) without symmetry and the coefficient support of the derived matrix P_e indeed can be controlled by that of P. The algorithms in [15, 19] for the special case $r = 1$ can be employed to handle a

general $r \times s$ matrix P without symmetry, see [15, 19] for detail. However, for the general case $r > 1$, it is no longer clear whether the coefficient support of the derived matrix P_e obtained by the algorithms in [15, 19] can still be controlled by that of P.

Several special cases of matrix extension with symmetry have been considered in the literature. For $\mathbb{F} = \mathbb{R}$ and $r = 1$, matrix extension with symmetry has been considered in [16]. For $r = 1$, matrix extension with symmetry has been studied in [7] and a simple algorithm is given there. In the context of wavelet analysis, several particular cases of matrix extension with symmetry related to the construction of wavelets and multiwavelets have been investigated in [2, 6, 7, 9, 13, 14, 16, 18]. However, for the general case of an $r \times s$ matrix, the approaches on matrix extension with symmetry in [7, 16] for the particular case $r = 1$ cannot be employed to handle the general case. The algorithms in [7, 16] are very difficult to be generalized to the general case $r > 1$, partially due to the complicated relations of the symmetry patterns between different rows of P. For the general case of matrix extension with symmetry, it becomes much harder to control the coefficient support of the derived matrix P_e, comparing with the special case $r = 1$. Extra effort is needed in any algorithm of deriving P_e so that its coefficient support can be controlled by that of P.

The contributions of this paper lie in the following aspects. Firstly, we satisfactorily solve the general matrix extension problem with symmetry for any r, s such that $1 \leqslant r \leqslant s$. More importantly, we obtain a complete representation of any $r \times s$ paraunitary matrix P having compatible symmetry with $1 \leqslant r \leqslant s$. This representation leads to a step-by-step algorithm for deriving a desired matrix P_e from a given matrix P. Secondly, we obtain an optimal result in the sense of (1.7) on controlling the coefficient support of the desired matrix P_e derived from a given matrix P by our algorithm. This is of importance in both theory and application, since short support of a filter or a multiwavelet is a highly desirable property and short support usually means a fast algorithm and simple implementation in practice. Thirdly, we introduce the notion of compatibility of symmetry, which plays a critical role in the study of the general matrix extension problem with symmetry for the multi-row case ($r \geqslant 1$). Fourthly, we provide a complete analysis and a systematic construction algorithm for d-band symmetric filter banks and symmetric orthonormal multiwavelets. Finally, most of the literature on the matrix extension problem only consider Laurent polynomials with coefficients in the special field $\mathbb{C}$ ([15]) or $\mathbb{R}$ ([1, 16]). In this paper, our setting is under a general field $\mathbb{F}$, which can be any subfield of $\mathbb{C}$ satisfying (1.6).

The structure of this paper is as follows. In section 2, we shall present a step-by-step algorithm which leads to constructive proofs of Theorems 1 and 2. In section 3, we shall discuss an application of our main results on matrix extension with symmetry to the design of symmetric filter banks in electronic engineering and to the construction of symmetric orthonormal multiwavelets in wavelet analysis. Examples will be provided to illustrate our algorithms. Finally, we shall prove Theorems 1 and 2 in section 4.

2. An Algorithm for Matrix Extension with Symmetry. In this section, we present a step-by-step algorithm on matrix extension with symmetry to derive a desired matrix P_e in Theorem 2 from a given matrix P. Our algorithm has three steps: initialization, support reduction, and finalization. The step of initialization reduces the symmetry pattern of P to a standard form. The step of support reduction is the main body of the algorithm, producing a sequence of elementary matrices $\mathsf{A}_1, \dots, \mathsf{A}_J$ that reduce the length of the coefficient support of P to 0. The step of finalization generates the desired matrix P_e as in Theorem 2. More precisely, our algorithm

written in the form of *pseudo-code* for Theorem 2 is as follows:

ALGORITHM 1. *Input* P *as in Theorem 2 with* $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^*\mathcal{S}\theta_2$ *for some* $1\times r$ *and* $1\times s$ *row vectors* θ_1 *and* θ_2 *of Laurent polynomials with symmetry.*

1. Initialization: *Let* $\mathsf{Q} := \mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$. *Then the symmetry pattern of* Q *is*

$$\mathcal{S}\mathsf{Q} = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}], \tag{2.1}$$

where all nonnegative integers $r_1,\ldots,r_4,s_1,\ldots,s_4$ *are uniquely determined by* $\mathcal{S}\mathsf{P}$.

2. Support Reduction: *Let* $\mathsf{P}_0 := \mathsf{U}^*_{\mathcal{S}\theta_2}$ *and* $J := 1$.

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while (|coeffsupp(Q)| > 0) do      %% outer while loop
  Let Q_0 := Q, [k_1, k_2] := coeffsupp(Q), and A_J := I_s.
  if k_2 = -k_1 then
    for j from 1 to r do
      Let q := [Q_0]_{j,:} and p := [Q]_{j,:}, the jth rows of Q_0 and Q, respectively.
      Let [ℓ_1, ℓ_2] := coeffsupp(q), ℓ := ℓ_2 - ℓ_1, and B_j := I_s.
      if coeffsupp(q) = coeffsupp(p) and ℓ ⩾ 2 and (ℓ_1 = k_1 or ℓ_2 = k_2) then
        B_j := B_q. A_J := A_J B_j. Q_0 := Q_0 B_j.
      end if
    end for
    Q_0 takes the form in (2.8).
    Let B_{(-k_2,k_2)} := I_s, Q_1 := Q_0, j_1 := 1 and j_2 := r_3 + r_4 + 1.
    while j_1 ⩽ r_1 + r_2 and j_2 ⩽ r do      %% inner while loop
      Let q_1 := [Q_1]_{j_1,:} and q_2 := [Q_1]_{j_2,:}.
      if coeff(q_1, k_1) = 0 then j_1 := j_1 + 1. end if
      if coeff(q_2, k_2) = 0 then j_2 := j_2 + 1. end if
      if coeff(q_1, k_1) ≠ 0 and coeff(q_2, k_2) ≠ 0 then
        B_{(-k_2,k_2)} := B_{(-k_2,k_2)} B_{(q_1,q_2)}. Q_1 := Q_1 B_{(q_1,q_2)}. A_J := A_J B_{(q_1,q_2)}.
        j_1 := j_1 + 1. j_2 := j_2 + 1.
      end if
    end while      %% end inner while loop
  end if
  Q_1 takes the form in (2.8) with either coeff(Q_1, -k) = 0 or coeff(Q_1, k) = 0.
  Let A_J := A_J B_{Q_1} and Q := Q A_J.
  Then SQ = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4}]^T [1_{s'_1}, -1_{s'_2}, z^{-1}1_{s'_3}, -z^{-1}1_{s'_4}].
  Replace s_1, ..., s_4 by s'_1, ..., s'_4, respectively. Let P_J := A_J^* and J := J + 1.
end while      %% end outer while loop
```

3. Finalization: $\mathsf{Q} = \mathrm{diag}(F_1,F_2,F_3,F_4)$ *for some* $r_j\times s_j$ *constant matrices* F_j *in* $\mathbb{F}$, $j=1,\ldots,4$. *Let* $U := \mathrm{diag}(U_{F_1},U_{F_2},U_{F_3},U_{F_4})$ *so that* $\mathsf{Q}U = [I_r, \mathbf{0}]$. *Define* $\mathsf{P}_J := U^*$ *and* $\mathsf{P}_{J+1} := \mathrm{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$.

Output a desired matrix P_e *satisfying all the properties in Theorem 2.*

In the following subsections, we present detailed constructions of the matrices $\mathsf{U}_{\mathcal{S}\theta}$, B_q, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{\mathsf{Q}_1}$, and U_F appearing in Algorithm 1.

2.1. Initialization. Let θ be a $1\times n$ row vector of Laurent polynomials with symmetry such that $\mathcal{S}\theta = [\varepsilon_1 z^{c_1},\ldots,\varepsilon_n z^{c_n}]$ for some $\varepsilon_1,\ldots,\varepsilon_n \in \{-1,1\}$ and $c_1,\ldots,c_n \in \mathbb{Z}$. Then, the symmetry of any entry in the vector $\theta\mathrm{diag}(z^{-\lceil c_1/2\rceil},\ldots,z^{-\lceil c_n/2\rceil})$ belongs to $\{\pm 1, \pm z^{-1}\}$. Thus, there is a permutation matrix E_θ to regroup these four types of symmetries together so that

$$\mathcal{S}(\theta\mathsf{U}_{\mathcal{S}\theta}) = [\mathbf{1}_{n_1}, -\mathbf{1}_{n_2}, z^{-1}\mathbf{1}_{n_3}, -z^{-1}\mathbf{1}_{n_4}], \tag{2.2}$$

where $\mathsf{U}_{\mathcal{S}\theta} := \operatorname{diag}(z^{-\lceil c_1/2 \rceil}, \dots, z^{-\lceil c_n/2 \rceil})E_\theta$, $\mathbf{1}_m$ denotes the $1 \times m$ row vector $[1, \dots, 1]$, and $n_1, \dots, n_4$ are nonnegative integers uniquely determined by $\mathcal{S}\theta$. Since P satisfies (1.4), it is easy to see that $\mathsf{Q} := \mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ has the symmetry pattern as in (2.1). Note that $\mathsf{U}_{\mathcal{S}\theta_1}$ and $\mathsf{U}_{\mathcal{S}\theta_2}$ do not increase the length of the coefficient support of P.

2.2. Support Reduction. Denote $\mathsf{Q} := \mathsf{U}^*_{\mathcal{S}\theta_1}\mathsf{P}\mathsf{U}_{\mathcal{S}\theta_2}$ as in Algorithm 1. The outer `while` loop in the step of support reduction produces a sequence of elementary paraunitary matrices $\mathsf{A}_1, \dots, \mathsf{A}_J$ that reduce the length of the coefficient support of Q gradually to 0. The construction of each A_j has three parts: $\{\mathsf{B}_1, \dots, \mathsf{B}_r\}$, $\mathsf{B}_{(-k,k)}$, and $\mathsf{B}_{\mathsf{Q}_1}$. The first part $\{\mathsf{B}_1, \dots, \mathsf{B}_r\}$ (see the `for` loop) is constructed recursively for each of the r rows of Q so that $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ has a special form as in (2.8). If both $\operatorname{coeff}(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and $\operatorname{coeff}(\mathsf{Q}_0, k) \neq \mathbf{0}$, then the second part $\mathsf{B}_{(-k,k)}$ (see the inner `while` loop) is further constructed so that $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ takes the form in (2.8) with at least one of $\operatorname{coeff}(\mathsf{Q}_1, -k)$ and $\operatorname{coeff}(\mathsf{Q}_1, k)$ being $\mathbf{0}$. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed to handle the case that $\operatorname{coeffsupp}(\mathsf{Q}_1) = [-k, k-1]$ or $\operatorname{coeffsupp}(\mathsf{Q}_1) = [-k+1, k]$ so that $\operatorname{coeffsupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1, k-1]$.

Let q denote an arbitrary row of Q with $|\operatorname{coeffsupp}(\mathsf{q})| \geqslant 2$. We first explain how to construct B_q for a given row q such that B_q reduces the length of the coefficient support of q by 2 and keeps its symmetry pattern. Note that in the `for` loop, B_j is simply B_q with q being the current jth row of $\mathsf{Q}\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$, where $\mathsf{B}_0 := I_s$.

By (2.1), we have $\mathcal{S}\mathsf{q} = \varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1, 1\}$ and $c \in \{0, 1\}$. For $\varepsilon = -1$, there is a permutation matrix E_ε such that $\mathcal{S}(\mathsf{q}E_\varepsilon) = z^c[\mathbf{1}_{s_2}, -\mathbf{1}_{s_1}, z^{-1}\mathbf{1}_{s_4}, -z^{-1}\mathbf{1}_{s_3}]$. For $\varepsilon = 1$, we let $E_\varepsilon := I_s$. Then, $\mathsf{q}E_\varepsilon$ must take the form in either (2.3) or (2.4) with $\mathtt{f}_1 \neq \mathbf{0}$ as follows:

$$\begin{aligned} \mathsf{q}E_\varepsilon = & [\mathtt{f}_1, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{\ell_1} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \operatorname{coeff}(\mathsf{q}E_\varepsilon, \ell)z^\ell \\ & + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell_2-1} + [\mathtt{f}_1, \mathtt{f}_2, \mathbf{0}, \mathbf{0}]z^{\ell_2}; \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathsf{q}E_\varepsilon = & [\mathbf{0}, \mathbf{0}, \mathtt{f}_1, -\mathtt{f}_2]z^{\ell_1} + [\mathtt{g}_1, -\mathtt{g}_2, \mathtt{f}_3, -\mathtt{f}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \operatorname{coeff}(\mathsf{q}E_\varepsilon, \ell)z^\ell \\ & + [\mathtt{g}_3, \mathtt{g}_4, \mathtt{f}_3, \mathtt{f}_4]z^{\ell_2-1} + [\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]z^{\ell_2}. \end{aligned} \tag{2.4}$$

If $\mathsf{q}E_\varepsilon$ takes the form in (2.4), we further construct a permutation matrix E_q such that $[\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]E_\mathsf{q} = [\mathtt{f}_1, \mathtt{f}_2, \mathtt{g}_1, \mathtt{g}_2]$ and define $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon E_\mathsf{q} \operatorname{diag}(I_{s-s_\mathtt{g}}, z^{-1}I_{s_\mathtt{g}})$, where $s_\mathtt{g}$ is the size of the row vector $[\mathtt{g}_1, \mathtt{g}_2]$. Then $\mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ takes the form in (2.3). For $\mathsf{q}E_\varepsilon$ of form (2.3), we simply let $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon$. In this way, $\mathsf{q}_0 := \mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ always takes the form in (2.3) with $\mathtt{f}_1 \neq \mathbf{0}$.

Note that $\mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{U}^*_{\mathsf{q},\varepsilon} = I_s$ and $\|\mathtt{f}_1\| = \|\mathtt{f}_2\|$ if $\mathsf{q}_0\mathsf{q}_0^* = 1$, where $\|\mathtt{f}\| := \sqrt{\mathtt{f}\mathtt{f}^*}$. Now we construct an $s \times s$ paraunitary matrix $\mathsf{B}_{\mathsf{q}_0}$ to reduce the coefficient support of q_0

as in (2.3) from $[\ell_1, \ell_2]$ to $[\ell_1+1, \ell_2-1]$ as follows:

$$\mathsf{B}^*_{\mathsf{q}_0} := \frac{1}{c}\left[\begin{array}{c|c|c|c} \mathtt{f}_1(z+\frac{c_0}{c_{\mathtt{f}_1}}+\frac{1}{z}) & \mathtt{f}_2(z-\frac{1}{z}) & \mathsf{g}_1(1+\frac{1}{z}) & \mathsf{g}_2(1-\frac{1}{z}) \\ cF_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline -\mathtt{f}_1(z-\frac{1}{z}) & -\mathtt{f}_2(z-\frac{c_0}{c_{\mathtt{f}_1}}+\frac{1}{z}) & -\mathsf{g}_1(1-\frac{1}{z}) & -\mathsf{g}_2(1+\frac{1}{z}) \\ \mathbf{0} & cF_2 & \mathbf{0} & \mathbf{0} \\ \hline \frac{c_{\mathsf{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1+z) & -\frac{c_{\mathsf{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1-z) & c_{\mathsf{g}'_1}\mathsf{g}'_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & cG_1 & \mathbf{0} \\ \hline \frac{c_{\mathsf{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1-z) & -\frac{c_{\mathsf{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1+z) & \mathbf{0} & c_{\mathsf{g}'_2}\mathsf{g}'_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & cG_2 \end{array}\right], \tag{2.5}$$

where $c_{\mathtt{f}_1} := \|\mathtt{f}_1\|$, $c_{\mathsf{g}_1} := \|\mathsf{g}_1\|$, $c_{\mathsf{g}_2} := \|\mathsf{g}_2\|$, $c_0 := \operatorname{coeff}(\mathsf{q}_0, \ell_1+1)\operatorname{coeff}(\mathsf{q}_0^*, -\ell_2)/c_{\mathtt{f}_1}$,

$$\begin{aligned} & c_{\mathsf{g}'_1} := \begin{cases} \frac{-2c_{\mathtt{f}_1}-\overline{c_0}}{c_{\mathsf{g}_1}} & \text{if } \mathsf{g}_1 \neq \mathbf{0}; \\ c & \text{otherwise,} \end{cases} \qquad c_{\mathsf{g}'_2} := \begin{cases} \frac{2c_{\mathtt{f}_1}-\overline{c_0}}{c_{\mathsf{g}_2}} & \text{if } \mathsf{g}_2 \neq \mathbf{0}; \\ c & \text{otherwise,} \end{cases} \\ & c := (4c_{\mathtt{f}_1}^2 + 2c_{\mathsf{g}_1}^2 + 2c_{\mathsf{g}_2}^2 + |c_0|^2)^{1/2}, \end{aligned} \tag{2.6}$$

and $[\frac{\mathtt{f}_j^*}{\|\mathtt{f}_j\|}, F_j^*] = U_{\mathtt{f}_j}$, $[\mathsf{g}'^*_j, G_j^*] = U_{\mathsf{g}_j}$ for $j = 1, 2$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\mathtt{f}_j, \mathsf{g}_j$ in $\mathbb{F}$, respectively (see section 4 for a concrete construction of such unitary matrices $U_{\mathtt{f}_j}$ and U_{g_j}). Here, the role of a unitary constant matrix $U_{\mathtt{f}}$ in $\mathbb{F}$ is to reduce the number of nonzero entries in $\mathtt{f}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \ldots, 0]$. The operations for the emptyset $\emptyset$ are defined by $\|\emptyset\| = \emptyset$, $\emptyset + A = A$ and $\emptyset \cdot A = \emptyset$ for any object A.

Define $\mathsf{B}_{\mathsf{q}} := \mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{B}_{\mathsf{q}_0}\mathsf{U}^*_{\mathsf{q},\varepsilon}$. Then B_{q} is paraunitary. Due to the particular form of $\mathsf{B}_{\mathsf{q}_0}$ as in (2.5), direct computations yield the following very important properties of the paraunitary matrix B_{q}:

(P1) $\mathcal{S}\mathsf{B}_{\mathsf{q}} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, $\operatorname{coeffsupp}(\mathsf{B}_{\mathsf{q}}) = [-1, 1]$, and $\operatorname{coeffsupp}(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = [\ell_1+1, \ell_2-1]$. That is, B_{q} has compatible symmetry with coefficient support on $[-1, 1]$ and B_{q} reduces the length of the coefficient support of q exactly by 2. Moreover, $\mathcal{S}(\mathsf{q}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}\mathsf{q}$.

(P2) if $(\mathsf{p}, \mathsf{q}^*)$ has mutually compatible symmetry and $\mathsf{p}\mathsf{q}^* = 0$, then $\mathcal{S}(\mathsf{p}\mathsf{B}_{\mathsf{q}}) = \mathcal{S}(\mathsf{p})$ and $\operatorname{coeffsupp}(\mathsf{p}\mathsf{B}_{\mathsf{q}}) \subseteq \operatorname{coeffsupp}(\mathsf{p})$. That is, B_{q} keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

Next, let us explain the construction of $\mathsf{B}_{(-k,k)}$. For $\operatorname{coeffsupp}(\mathsf{Q}) = [-k, k]$ with $k \geqslant 1$, Q is of the form as follows:

$$\begin{aligned} \mathsf{Q} = & \left[\begin{array}{cccc} F_{11} & -F_{21} & G_{31} & -G_{41} \\ -F_{12} & F_{22} & -G_{32} & G_{42} \\ \hline \mathbf{0} & \mathbf{0} & F_{31} & -F_{41} \\ \mathbf{0} & \mathbf{0} & -F_{32} & F_{42} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} F_{51} & -F_{61} & G_{71} & -G_{81} \\ -F_{52} & F_{61} & -G_{72} & G_{82} \\ \hline G_{11} & -G_{21} & F_{71} & -F_{81} \\ -G_{12} & G_{22} & -F_{72} & F_{82} \end{array}\right] z^{-k+1} \\ & + \sum_{n=2-k}^{k-2} \operatorname{coeff}(\mathsf{Q}, n) + \left[\begin{array}{cccc} F_{51} & F_{61} & G_{31} & G_{41} \\ F_{52} & F_{61} & G_{32} & G_{42} \\ \hline G_{51} & G_{61} & F_{71} & F_{81} \\ G_{52} & G_{62} & F_{72} & F_{82} \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} F_{11} & F_{21} & \mathbf{0} & \mathbf{0} \\ F_{12} & F_{22} & \mathbf{0} & \mathbf{0} \\ \hline G_{11} & G_{21} & F_{31} & F_{41} \\ G_{12} & G_{22} & F_{32} & F_{42} \end{array}\right] z^{k} \end{aligned} \tag{2.7}$$

with all F_{jk}'s and G_{jk}'s being constant matrices in $\mathbb{F}$ and $F_{11}, F_{22}, F_{31}, F_{42}$ being of size $r_1 \times s_1$, $r_2 \times s_2$, $r_3 \times s_3$, $r_4 \times s_4$, respectively. Due to Property (P1) and (P2) of B_{q}, the `for` loop in Algorithm 1 reduces Q in (2.7) to $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1 \cdots \mathsf{B}_r$ as follows:

$$\left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \widetilde{G}_{31} & -\widetilde{G}_{41} \\ \mathbf{0} & \mathbf{0} & -\widetilde{G}_{32} & \widetilde{G}_{42} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \cdots + \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{G}_{11} & \widetilde{G}_{21} & \mathbf{0} & \mathbf{0} \\ \widetilde{G}_{12} & \widetilde{G}_{22} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k}. \tag{2.8}$$

If either $\mathrm{coeff}(\mathsf{Q}_0, -k) = \mathbf{0}$ or $\mathrm{coeff}(\mathsf{Q}_0, k) = \mathbf{0}$, then the inner `while` loop does nothing and $\mathsf{B}_{(-k,k)} = I_s$. If both $\mathrm{coeff}(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and $\mathrm{coeff}(\mathsf{Q}_0, k) \neq \mathbf{0}$, then $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1, \mathsf{q}_2)$ with $\mathsf{q}_1, \mathsf{q}_2$ being two rows of Q_0 satisfying $\mathrm{coeff}(\mathsf{q}_1, -k) \neq \mathbf{0}$ and $\mathrm{coeff}(\mathsf{q}_2, k) \neq \mathbf{0}$. The construction of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ with respect to such a pair $(\mathsf{q}_1, \mathsf{q}_2)$ in the inner `while` loop is as follows.

Similar to the discussion before (2.3), there is a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ such that $\widetilde{\mathsf{q}}_1 := \mathsf{q}_1 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ and $\widetilde{\mathsf{q}}_2 := \mathsf{q}_2 E_{(\mathsf{q}_1,\mathsf{q}_2)}$ take the following form:

$$
\begin{aligned}
\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix} &= \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \widetilde{g}_3 & -\widetilde{g}_4 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} \widetilde{f}_5 & -\widetilde{f}_6 & \widetilde{g}_7 & -\widetilde{g}_8 \\ \hline \varepsilon\widetilde{g}_1 & -\varepsilon\widetilde{g}_2 & \varepsilon\widetilde{f}_7 & -\varepsilon\widetilde{f}_8 \end{array}\right] z^{-k+1} \\
&\quad + \sum_{n=2-k}^{k-2} \mathrm{coeff}\Big(\begin{bmatrix} \widetilde{\mathsf{q}}_1 \\ \widetilde{\mathsf{q}}_2 \end{bmatrix}, n\Big) + \left[\begin{array}{cccc} \widetilde{f}_5 & \widetilde{f}_6 & \widetilde{g}_3 & \widetilde{g}_4 \\ \hline \widetilde{g}_5 & \widetilde{g}_6 & \widetilde{f}_7 & \widetilde{f}_8 \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \widetilde{g}_1 & \widetilde{g}_2 & \mathbf{0} & \mathbf{0} \end{array}\right] z^{k},
\end{aligned} \tag{2.9}
$$

where $\varepsilon \in \{-1, 1\}$ and all $\widetilde{\mathsf{g}}_j$'s are nonzero row vectors. Note that $\|\widetilde{\mathsf{g}}_1\| = \|\widetilde{\mathsf{g}}_2\| =: c_{\widetilde{\mathsf{g}}_1}$ and $\|\widetilde{\mathsf{g}}_3\| = \|\widetilde{\mathsf{g}}_4\| =: c_{\widetilde{\mathsf{g}}_3}$. Construct an $s \times s$ paraunitary matrix $\mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)}$ as follows:

$$
\mathsf{B}^*_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)} := \frac{1}{c} \left[\begin{array}{c|c|c|c}
\frac{c_0}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1 & \mathbf{0} & \widetilde{\mathsf{g}}_3(1+\frac{1}{z}) & \widetilde{\mathsf{g}}_4(1-\frac{1}{z}) \\
c\widetilde{G}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline
\mathbf{0} & \frac{c_0}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2 & -\widetilde{\mathsf{g}}_3(1-\frac{1}{z}) & -\widetilde{\mathsf{g}}_4(1+\frac{1}{z}) \\
\mathbf{0} & c\widetilde{G}_2 & \mathbf{0} & \mathbf{0} \\ \hline
\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1(1+z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2(1-z) & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}}\widetilde{\mathsf{g}}_3 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & c\widetilde{G}_3 & \mathbf{0} \\ \hline
\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1(1-z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2(1+z) & \mathbf{0} & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}}\widetilde{\mathsf{g}}_4 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & c\widetilde{G}_4
\end{array}\right], \tag{2.10}
$$

where $c_0 := \mathrm{coeff}(\widetilde{\mathsf{q}}_1, -k+1)\mathrm{coeff}(\widetilde{\mathsf{q}}_2^*, -k)/c_{\widetilde{\mathsf{g}}_1}$, $c := (|c_0|^2 + 4c_{\widetilde{\mathsf{g}}_3}^2)^{1/2}$, and $[\frac{\widetilde{\mathsf{g}}_j^*}{\|\widetilde{\mathsf{g}}_j\|}, \widetilde{G}_j^*] = U_{\widetilde{\mathsf{g}}_j}$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\widetilde{\mathsf{g}}_j$ in $\mathbb{F}$, $j = 1, \dots, 4$, respectively. Let $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} := E_{(\mathsf{q}_1,\mathsf{q}_2)} \mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)} E^T_{(\mathsf{q}_1,\mathsf{q}_2)}$. Similar to Property (P1) and (P2) of B_q, we have the following very important properties of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$:

(P3) $\mathcal{S}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, the coefficient support of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ is on $[-1, 1]$, $\mathrm{coeffsupp}(\mathsf{q}_1 \mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$ and $\mathrm{coeffsupp}(\mathsf{q}_2 \mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-k+1, k-1]$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ has compatible symmetry with coefficient support on $[-1, 1]$ and $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ reduces the length of both the coefficient supports of q_1 and q_2 by 2. Moreover, $\mathcal{S}(\mathsf{q}_1 \mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_1$ and $\mathcal{S}(\mathsf{q}_2 \mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{q}_2$.

(P4) if both $(\mathsf{p}, \mathsf{q}_1^*)$ and $(\mathsf{p}, \mathsf{q}_2^*)$ have mutually compatible symmetry and $\mathsf{p}\mathsf{q}_1^* = \mathsf{p}\mathsf{q}_2^* = 0$, then $\mathcal{S}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathcal{S}\mathsf{p}$ and $\mathrm{coeffsupp}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq \mathrm{coeffsupp}(\mathsf{p})$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

Now, due to the Property (P3) and (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{(-k,k)}$ constructed in the inner `while` loop reduces Q_0 of the form in (2.8) with both $\mathrm{coeff}(\mathsf{Q}_0, -k) \neq \mathbf{0}$ and $\mathrm{coeff}(\mathsf{Q}_0, k) \neq \mathbf{0}$, to $\mathsf{Q}_1 := \mathsf{Q}_0 \mathsf{B}_{(-k,k)}$ of the form in (2.8) with either $\mathrm{coeff}(\mathsf{Q}_1, -k) = \mathrm{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$ (for this case, simply let $\mathsf{B}_{\mathsf{Q}_1} := I_s$) or one of $\mathrm{coeff}(\mathsf{Q}_1, -k)$ and $\mathrm{coeff}(\mathsf{Q}_1, k)$ is nonzero. For the latter case, $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(U_1 \mathsf{W}_1, I_{s_3+s_4})E$ with matrices U_1, W_1 constructed with respect to $\mathrm{coeff}(\mathsf{Q}_1, k) \neq \mathbf{0}$ or $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(I_{s_1+s_2}, U_3 \mathsf{W}_3)E$ with U_3, W_3 constructed with respect to $\mathrm{coeff}(\mathsf{Q}_1, -k) \neq \mathbf{0}$, where E is a permutation

matrix. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed so that coeffsupp$(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1, k-1]$. Let Q_1 take form in (2.8). The matrices U_1, W_1 or U_3, W_3, and E are constructed as follows.

Let $U_1 := \mathrm{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and $U_3 := \mathrm{diag}(U_{\widetilde{G}_3}, U_{\widetilde{G}_4})$ with

$$\widetilde{G}_1 := \begin{bmatrix} \widetilde{G}_{11} \\ \widetilde{G}_{12} \end{bmatrix}, \widetilde{G}_2 := \begin{bmatrix} \widetilde{G}_{21} \\ \widetilde{G}_{22} \end{bmatrix}, \widetilde{G}_3 := \begin{bmatrix} \widetilde{G}_{31} \\ \widetilde{G}_{32} \end{bmatrix}, \widetilde{G}_4 := \begin{bmatrix} \widetilde{G}_{41} \\ \widetilde{G}_{42} \end{bmatrix}. \tag{2.11}$$

Here, for a nonzero matrix G with rank m, U_G is a unitary matrix such that $GU_G = [R, \mathbf{0}]$ for some matrix R of rank m. For $G = \mathbf{0}, U_G := I$ and for $G = \emptyset, U_G := \emptyset$. When $G_1G_1^* = G_2G_2^*$, U_{G_1} and U_{G_2} can be constructed such that $G_1U_{G_1} = [R, \mathbf{0}]$ and $G_2U_{G_2} = [R, \mathbf{0}]$ (see section 4 for more detail).

Let m_1, m_3 be the ranks of $\widetilde{G}_1$, $\widetilde{G}_3$, respectively ($m_1 = 0$ when coeff$(\mathsf{Q}_1, k) = \mathbf{0}$ and $m_3 = 0$ when coeff$(\mathsf{Q}_1, -k) = \mathbf{0}$). Note that $\widetilde{G}_1\widetilde{G}_1^* = \widetilde{G}_2\widetilde{G}_2^*$ or $\widetilde{G}_3\widetilde{G}_3^* = \widetilde{G}_4\widetilde{G}_4^*$ due to $\mathsf{Q}_1\mathsf{Q}_1^* = I_r$. The matrices $\mathsf{W}_1, \mathsf{W}_3$ are then constructed by:

$$\mathsf{W}_1 := \left[\begin{array}{c|c|c|c} \mathsf{U}_1 & & \mathsf{U}_2 & \\ \hline & I_{s_1-m_1} & & \\ \hline \mathsf{U}_2 & & \mathsf{U}_1 & \\ \hline & & & I_{s_2-m_1} \end{array}\right], \mathsf{W}_3 := \left[\begin{array}{c|c|c|c} \mathsf{U}_3 & & \mathsf{U}_4 & \\ \hline & I_{s_3-m_3} & & \\ \hline \mathsf{U}_4 & & \mathsf{U}_3 & \\ \hline & & & I_{s_4-m_3} \end{array}\right], \tag{2.12}$$

where $\mathsf{U}_1(z) = -\mathsf{U}_2(-z) := \frac{1+z^{-1}}{2} I_{m_1}$ and $\mathsf{U}_3(z) = \mathsf{U}_4(-z) := \frac{1+z}{2} I_{m_3}$.

Let $\mathsf{W}_{\mathsf{Q}_1} := \mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})$ for the case that coeff$(\mathsf{Q}_1, k) \neq \mathbf{0}$ or $\mathsf{W}_{\mathsf{Q}_1} := \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)$ for the case that coeff$(\mathsf{Q}_1, -k) \neq \mathbf{0}$. Then $\mathsf{W}_{\mathsf{Q}_1}$ is paraunitary. By the symmetry pattern and orthogonality of Q_1, $\mathsf{W}_{\mathsf{Q}_1}$ reduces the coefficient support of Q_1 to $[-k+1, k-1]$, i.e., coeffsupp$(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [-k+1, k-1]$. Moreover, $\mathsf{W}_{\mathsf{Q}_1}$ changes the symmetry pattern of Q_1 such that $\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4}]^T \mathcal{S}\theta_1$ with

$$\mathcal{S}\theta_1 = [z^{-1}\mathbf{1}_{m_1}, \mathbf{1}_{s_1-m_1}, -z^{-1}\mathbf{1}_{m_1}, -\mathbf{1}_{s_2-m_1}, \mathbf{1}_{m_3}, z^{-1}\mathbf{1}_{s_3-m_3}, -\mathbf{1}_{m_3}, -z^{-1}\mathbf{1}_{s_4-m_3}].$$

E is then the permutation matrix such that

$$\mathcal{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1})E = [\mathbf{1}_{r_1}, -\mathbf{1}_{r_2}, z\mathbf{1}_{r_3}, -z\mathbf{1}_{r_4},]^T \mathcal{S}\theta,$$

with $\mathcal{S}\theta = [\mathbf{1}_{s_1-m_1+m_3},, -\mathbf{1}_{s_2-m_1+m_3}, z^{-1}\mathbf{1}_{s_3-m_3+m_1}, -z^{-1}\mathbf{1}_{s_4-m_3+m_1}] = (\mathcal{S}\theta_1)E$.

3. Application to Filter Banks and Orthonormal Multiwavelets with Symmetry. In this section, we shall discuss the application of our results on matrix extension with symmetry to d-band symmetric paraunitary filter banks in electronic engineering and to orthonormal multiwavelets with symmetry in wavelet analysis. In order to do so, let us introduce some definitions first.

We say that d is *a dilation factor* if d is an integer with $|\mathsf{d}| > 1$. Throughout this section, d denotes a dilation factor. For simplicity of presentation, we further assume that d is positive, while multiwavelets and filter banks with a negative dilation factor can be handled similarly by a slight modification of the statements in this paper.

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that (1.6) holds. A low-pass filter $a_0 : \mathbb{Z} \mapsto \mathbb{F}^{r\times r}$ with multiplicity r is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}$. The *symbol* of the filter a_0 is defined to be $\mathsf{a}_0(z) := \sum_{k\in\mathbb{Z}} a_0(k)z^k$, which is a matrix of Laurent polynomials with coefficients in $\mathbb{F}$. Moreover, the d*-band subsymbols* of a_0 are defined by $\mathsf{a}_{0;\gamma}(z) := \sqrt{\mathsf{d}} \sum_{k\in\mathbb{Z}} a_0(\gamma + \mathsf{d}k)z^k$, $\gamma \in \mathbb{Z}$. We say that a_0 (or a_0) is *a* d*-band orthogonal filter* if

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}^*_{0;\gamma}(z) = I_r, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{3.1}$$

To design an orthogonal filter bank with the perfect reconstruction property, one has to design high-pass filters $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \mapsto \mathbb{F}^{r\times r}$ such that the polyphase matrix

$$\mathcal{P}(z) = \begin{bmatrix} \mathsf{a}_{0;0}(z) & \cdots & \mathsf{a}_{0;\mathsf{d}-1}(z) \\ \mathsf{a}_{1;0}(z) & \cdots & \mathsf{a}_{1;\mathsf{d}-1}(z) \\ \vdots & \vdots & \vdots \\ \mathsf{a}_{\mathsf{d}-1;0}(z) & \cdots & \mathsf{a}_{\mathsf{d}-1;\mathsf{d}-1}(z) \end{bmatrix} \tag{3.2}$$

is paraunitary, that is, $\mathcal{P}(z)\mathcal{P}^*(z) = I_{\mathsf{d}r}$, where each $\mathsf{a}_{m;\gamma}$ is a subsymbol of a_m for $m, \gamma = 0, \dots, \mathsf{d}-1$, respectively. Symmetry of the filters in a filter bank is a very much desirable property in many applications. We say that the low-pass filter a_0 (or a_0) has symmetry if

$$\mathsf{a}_0(z) = \operatorname{diag}(\varepsilon_1 z^{\mathsf{d}c_1}, \dots, \varepsilon_r z^{\mathsf{d}c_r})\mathsf{a}_0(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}) \tag{3.3}$$

for some $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ and $c_1, \dots, c_r \in \mathbb{R}$ such that $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$. To design a symmetric filter bank with the perfect reconstruction property, from a given d-band orthogonal low-pass filter a_0, one has to construct high-pass filters $a_1, \dots, a_{\mathsf{d}-1} : \mathbb{Z} \mapsto \mathbb{F}^{r\times r}$ such that all of them have symmetry that is compatible with the symmetry of a_0 in (3.3) and the polyphase matrix $\mathcal{P}$ in (3.2) is paraunitary.

For $f \in L_1(\mathbb{R})$, the Fourier transform used in this paper is defined to be $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ and can be naturally extended to $L_2(\mathbb{R})$ functions. For a d-band orthogonal low-pass filter a_0, we assume that there exists *an orthogonal* d*-refinable function vector* $\phi = [\phi_1, \dots, \phi_r]^T$ associated with the low-pass filter a_0, with compactly supported functions $\phi_1, \dots, \phi_r$ in $L_2(\mathbb{R})$, such that

$$\hat{\phi}(\mathsf{d}\xi) = \mathsf{a}_0(e^{-i\xi})\hat{\phi}(\xi), \qquad \xi \in \mathbb{R} \qquad \text{with} \quad \|\hat{\phi}(0)\| = 1, \tag{3.4}$$

and

$$\langle \phi(\cdot - k), \phi \rangle := \int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}^T dx = \delta(k) I_r, \qquad k \in \mathbb{Z}, \tag{3.5}$$

where δ denotes the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$. Define multiwavelet function vectors $\psi^m = [\psi_1^m, \dots, \psi_r^m]^T$ associated with the high-pass filters a_m, $m = 1, \dots, \mathsf{d}-1$, by

$$\widehat{\psi^m}(\mathsf{d}\xi) := \mathsf{a}_m(e^{-i\xi})\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R},\ m = 1, \dots, \mathsf{d}-1. \tag{3.6}$$

It is well known that $\{\psi^1, \dots, \psi^{\mathsf{d}-1}\}$ generates an orthonormal multiwavelet basis in $L_2(\mathbb{R})$; that is, $\{\mathsf{d}^{j/2}\psi_\ell^m(\mathsf{d}^j \cdot -k) \ :\ j, k \in \mathbb{Z}; m = 1, \dots, \mathsf{d}-1; \ell = 1, \dots, r\}$ is an orthonormal basis of $L_2(\mathbb{R})$, for example, see [3, 8, 11, 17] and references therein.

If a_0 has symmetry as in (3.3) and if 1 is a simple eigenvalue of $\mathsf{a}_0(1)$, then it is well known that the d-refinable function vector ϕ in (3.4) associated with the low-pass filter a_0 has the following symmetry:

$$\phi_1(c_1 - \cdot) = \varepsilon_1\phi_1, \quad \phi_2(c_2 - \cdot) = \varepsilon_2\phi_2, \quad \dots, \quad \phi_r(c_r - \cdot) = \varepsilon_r\phi_r. \tag{3.7}$$

Under the symmetry condition in (3.3), to apply Theorem 1, we first show that there exists a suitable paraunitary matrix U acting on $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ so that $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry. Note that $\mathsf{P}_{\mathsf{a}_0}$ itself may not have any symmetry.

LEMMA 1. *Let* $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$*, where* $\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}$ *are* d*-band subsymbols of a* d*-band orthogonal filter* a_0 *satisfying* (3.3)*. Then there exists a* $\mathsf{d}r \times \mathsf{d}r$ *paraunitary matrix* U *such that* $\mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ *has compatible symmetry.*

Proof. From (3.3), we deduce that

$$[\mathsf{a}_{0;\gamma}(z)]_{\ell,j} = \varepsilon_\ell \varepsilon_j z^{R^\gamma_{\ell,j}} [\mathsf{a}_{0;Q^\gamma_{\ell,j}}(z^{-1})]_{\ell,j},\ \gamma = 0,\dots,\mathsf{d}-1; \ell, j = 1,\dots,r, \tag{3.8}$$

where $\gamma, Q^\gamma_{\ell,j} \in \Gamma := \{0,\dots,\mathsf{d}-1\}$ and $R^\gamma_{\ell,j}$, $Q^\gamma_{\ell,j}$ are uniquely determined by

$$\mathsf{d}c_\ell - c_j - \gamma = \mathsf{d}R^\gamma_{\ell,j} + Q^\gamma_{\ell,j} \quad \text{with} \quad R^\gamma_{\ell,j} \in \mathbb{Z},\ Q^\gamma_{\ell,j} \in \Gamma. \tag{3.9}$$

Since $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1,\dots,r$, we have $c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1,\dots,r$ and therefore, $Q^\gamma_{\ell,j}$ is independent of ℓ. Consequently, by (3.8), for every $1 \leqslant j \leqslant r$, the jth column of the matrix $\mathsf{a}_{0;\gamma}$ is a flipped version of the jth column of the matrix $\mathsf{a}_{0;Q^\gamma_{\ell,j}}$. Let $\kappa_{j,\gamma} \in \mathbb{Z}$ be an integer such that $|\mathrm{coeffsupp}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j})|$ is as small as possible. Define $\mathsf{P} := [\mathsf{b}_{0;0},\dots,\mathsf{b}_{0;\mathsf{d}-1}]$ as follows:

$$[\mathsf{b}_{0;\gamma}]_{:,j} := \begin{cases} [\mathsf{a}_{0;\gamma}]_{:,j}, & \gamma = Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma < Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} - z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma > Q^\gamma_{\ell,j}, \end{cases} \tag{3.10}$$

where $[\mathsf{a}_{0;\gamma}]_{:,j}$ denotes the jth column of $\mathsf{a}_{0;\gamma}$. Let U denote the unique transform matrix corresponding to (3.10) such that $\mathsf{P} := [\mathsf{b}_{0;0},\dots,\mathsf{b}_{0;\mathsf{d}-1}] = [\mathsf{a}_{0;0},\dots,\mathsf{a}_{0;\mathsf{d}-1}]\mathsf{U}$. It is evident that U is paraunitary and $\mathsf{P} = \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$. We now show that P has compatible symmetry. Indeed, by (3.8) and (3.10),

$$[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j} = \mathrm{sgn}(Q^\gamma_{\ell,j} - \gamma)\varepsilon_\ell \varepsilon_j z^{R^\gamma_{\ell,j} + \kappa_{j,\gamma}}, \tag{3.11}$$

where $\mathrm{sgn}(x) = 1$ for $x \geqslant 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. By (3.9) and noting that $Q^\gamma_{\ell,j}$ is independent of ℓ, we have

$$\frac{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,j}}{[\mathcal{S}\mathsf{b}_{0;\gamma}]_{n,j}} = \varepsilon_\ell \varepsilon_n z^{R^\gamma_{\ell,j} - R^\gamma_{n,j}} = \varepsilon_\ell \varepsilon_n z^{c_\ell - c_n},$$

for all $1 \leqslant \ell, n \leqslant r$, which is equivalent to saying that P has compatible symmetry. ∎

Now, for a d-band orthogonal low-pass filter a_0 satisfying (3.3), we have the following algorithm to construct high-pass filters $\mathsf{a}_1,\dots,\mathsf{a}_{\mathsf{d}-1}$ such that they form a symmetric paraunitary filter bank with the perfect reconstruction property.

ALGORITHM 2. *Input an orthogonal* d*-band filter* a_0 *with symmetry in* (3.3).

(1) *Construct* U *as in* (3.10) *such that* $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ *has compatible symmetry:* $\mathcal{S}\mathsf{P} = [\varepsilon_1 z^{k_1},\dots,\varepsilon_r z^{k_r}]^T \mathcal{S}\theta$ *for some* $k_1,\dots,k_r \in \mathbb{Z}$ *and some* $1 \times \mathsf{d}r$ *row vector* θ *of Laurent polynomials with symmetry.*
(2) *Derive* P_e *with all the properties as in Theorem 1 from* P *by Algorithm 1.*
(3) *Let* $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0 \leqslant m,\gamma \leqslant \mathsf{d}-1}$ *as in* (3.2). *Define high-pass filters*

$$\mathsf{a}_m(z) := \frac{1}{\sqrt{\mathsf{d}}} \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{m;\gamma}(z^\mathsf{d}) z^\gamma, \qquad m = 1,\dots,\mathsf{d}-1. \tag{3.12}$$

Output a symmetric filter bank $\{\mathsf{a}_0, \mathsf{a}_1,\dots,\mathsf{a}_{\mathsf{d}-1}\}$ *with the perfect reconstruction property, i.e.* $\mathcal{P}$ *in* (3.2) *is paraunitary and all filters* a_m, $m = 1,\dots,\mathsf{d}-1$, *have symmetry:*

$$\mathsf{a}_m(z) = \mathrm{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m},\dots,\varepsilon_r^m z^{\mathsf{d}c_r^m})\mathsf{a}_m(1/z)\,\mathrm{diag}(\varepsilon_1 z^{-c_1},\dots,\varepsilon_r z^{-c_r}), \tag{3.13}$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ *and all* $\varepsilon_\ell^m \in \{-1,1\}$, $k_\ell^m \in \mathbb{Z}$, *for* $\ell = 1,\dots,r$ *and* $m = 1,\dots,\mathsf{d}-1$, *are determined by the symmetry pattern of* P_e *as follows:*

$$[\varepsilon_1 z^{k_1},\dots,\varepsilon_r z^{k_r},\varepsilon_1^1 z^{k_1^1},\dots,\varepsilon_r^1 z^{k_r^1},\dots,z^{k_1^{\mathsf{d}-1}},\dots,\varepsilon_r^{\mathsf{d}-1} z^{k_r^{\mathsf{d}-1}}]^T \mathcal{S}\theta := \mathcal{S}\mathsf{P}_e. \qquad (3.14)$$

Proof. Rewrite $\mathsf{P}_e = (\mathsf{b}_{m;\gamma})_{0\leqslant m,\gamma\leqslant \mathsf{d}-1}$ as a $\mathsf{d}\times\mathsf{d}$ block matrix with $r\times r$ blocks $\mathsf{b}_{m;\gamma}$. Since P_e has compatible symmetry as in (3.14), we have $[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,:} = \varepsilon_\ell^m \varepsilon_\ell z^{k_\ell^m - k_\ell}[\mathcal{S}\mathsf{b}_{0;\gamma}]_{\ell,:}$ for $\ell = 1,\dots,r$ and $m = 1,\dots,\mathsf{d}-1$. By (3.11), we have

$$[\mathcal{S}\mathsf{b}_{m;\gamma}]_{\ell,j} = \operatorname{sgn}(Q_{\ell,j}^\gamma - \gamma)\varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + \kappa_{j,\gamma} + k_\ell^m - k_\ell}, \qquad \ell, j = 1,\dots,r. \qquad (3.15)$$

By (3.15) and the definition of U^* in (3.10), we deduce that

$$[\mathsf{a}_{m;\gamma}]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^\gamma + k_\ell^m - k_\ell}[\mathsf{a}_{m;Q_{\ell,j}^\gamma}(z^{-1})]_{\ell,j}. \qquad (3.16)$$

This implies that $[\mathcal{S}\mathsf{a}_m]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{\mathsf{d}(k_\ell^m - k_\ell + c_\ell) - c_j}$, which is equivalent to (3.13) with $c_\ell^m := k_\ell^m - k_\ell + c_\ell$ for $m = 1,\dots,\mathsf{d}-1$ and $\ell = 1,\dots,r$. ☐

Since the high-pass filters $\mathsf{a}_1,\dots,\mathsf{a}_{\mathsf{d}-1}$ satisfy (3.13), it is easy to verify that each $\psi^m = [\psi_1^m,\dots,\psi_r^m]^T$ defined in (3.6) also has the following symmetry:

$$\psi_1^m(c_1^m - \cdot) = \varepsilon_1^m \psi_1^m, \quad \psi_2^m(c_2^m - \cdot) = \varepsilon_2^m \psi_2^m, \quad \dots, \quad \psi_r^m(c_r^m - \cdot) = \varepsilon_r^m \psi_r^m. \quad (3.17)$$

In the following, let us present several examples to demonstrate our results and illustrate our algorithms.

Example 1. Let $\mathsf{d} = 2$ and $r = 2$. A 2-band orthogonal low-pass filter a_0 with multiplicity 2 in [5] is given by

$$\mathsf{a}_0(z) = \frac{1}{40}\begin{bmatrix} 12(1+z^{-1}) & 16\sqrt{2}z^{-1} \\ -\sqrt{2}(z^2 - 9z - 9 + z^{-1}) & -2(3z - 10 + 3z^{-1}) \end{bmatrix}.$$

The low-pass filter a_0 satisfies (3.3) with $c_1 = -1, c_2 = 0$ and $\varepsilon_1 = \varepsilon_2 = 1$. Using Lemma 1, we obtain $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0},\mathsf{a}_{0;1}]$ and U as follows:

$$\mathsf{P}_{\mathsf{a}_0} = \frac{1}{20}\left[\begin{array}{cc|cc} 6\sqrt{2} & 0 & \frac{6\sqrt{2}}{z} & \frac{16}{z} \\ 9-z & 10\sqrt{2} & 9-\frac{1}{z} & -3\sqrt{2}(1+\frac{1}{z}) \end{array}\right], \ \mathsf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ z & 0 & -z & 0 \\ 0 & 0 & 0 & \sqrt{2}z \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1,z]^T[1,z^{-1},-1,1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{2}}{20}\begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \end{bmatrix}.$$

Applying Algorithm 1, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{2}}{20}\begin{bmatrix} 6\sqrt{2} & 0 & 0 & 8\sqrt{2} \\ 4(1+z) & 10 & 5(1-z) & -3(1+z) \\ 4(1+z) & -10 & 5(1-z) & -3(1+z) \\ 4\sqrt{2}(1-z) & 0 & 5\sqrt{2}(z+1) & 3\sqrt{2}(z-1) \end{bmatrix}.$$

We have $\mathcal{S}\mathsf{P}_e = [1,z,z,-z]^T[1,z^{-1},-1,1]$ and $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 4$. Now, from the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\leqslant m,\gamma\leqslant 1}$, we derive a high-pass filter a_1 as follows:

$$\mathsf{a}_1(z) = \frac{1}{40}\begin{bmatrix} -\sqrt{2}(z^2 - 9z - 9 + z^{-1}) & -2(3z + 10 + 3z^{-1}) \\ 2(z^2 - 9z + 9 - z^{-1}) & 6\sqrt{2}(z - z^{-1}) \end{bmatrix}.$$

Then the high-pass filter a_1 satisfies (3.13) with $c_1^1 = c_2^1 = 0$ and $\varepsilon_1^1 = 1, \varepsilon_2^1 = -1$.

EXAMPLE 2. Let $\mathsf{d} = 3$ and $r = 2$. A 3-band orthogonal low-pass filter a_0 with multiplicity 2 in [11] is given by:

$$\mathsf{a}_0(z) = \frac{1}{540} \begin{bmatrix} a_{11}(z) + a_{11}(z^{-1}) & a_{12}(z) + z^{-1}a_{12}(z^{-1}) \\ a_{21}(z) + z^3 a_{21}(z^{-1}) & a_{22}(z) + z^2 a_{22}(z^{-1}) \end{bmatrix},$$

where

$$\begin{aligned}
a_{11}(z) &= 90 + (55 - 5\sqrt{41})z - (8 + 2\sqrt{41})z^2 + (7\sqrt{41} - 47)z^4; \\
a_{12}(z) &= 145 + 5\sqrt{41} + (1 - \sqrt{41})z^2 + (34 - 4\sqrt{41})z^3; \\
a_{21}(z) &= (111 + 9\sqrt{41})z^2 + (69 - 9\sqrt{41})z^4; \\
a_{22}(z) &= 90z + (63 - 3\sqrt{41})z^2 + (3\sqrt{41} - 63)z^3.
\end{aligned}$$

The low-pass filter a_0 satisfies (3.3) with $c_1 = 0, c_2 = 1$ and $\varepsilon_1 = \varepsilon_2 = 1$. From $\mathsf{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}, \mathsf{a}_{0;2}]$, the matrix U constructed by Lemma 1 is given by

$$\mathsf{U} := \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & z & 0 & -z & 0 \\ 0 & z & 0 & 0 & 0 & -z \end{bmatrix}.$$

Let

$$\begin{array}{llll}
c_0 = 11 - \sqrt{41}; & t_{12} = 5(7 - \sqrt{41}); & c_{12} = 10(29 + \sqrt{41}); & t_{13} = -5c_0; \\
t_{16} = 3c_0; & t_{15} = 3(3\sqrt{41} - 13); & t_{25} = 6(7 + 3\sqrt{41}); & t_{26} = 6(21 - \sqrt{41}); \\
t_{53} = 400\sqrt{6}/c_0; & t_{55} = 12\sqrt{6}(\sqrt{41} - 1); & t_{56} = 6\sqrt{6}(4 + \sqrt{41}); & c_{66} = 3\sqrt{6}(3 + 7\sqrt{41}).
\end{array}$$

Then $\mathsf{P} := \mathsf{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [1, z]^T [1, 1, 1, z^{-1}, -1, -1]$ and is given by

$$\mathsf{P} = \frac{\sqrt{6}}{1080} \begin{bmatrix} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z - z^{-1}) & t_{16}(z - z^{-1}) \\ 0 & 0 & 180(1 + z) & 180\sqrt{2} & t_{25}(1 - z) & t_{26}(1 - z) \end{bmatrix},$$

where $b_{12}(z) = t_{12}(z + z^{-1}) + c_{12}$ and $b_{13}(z) = t_{13}(z - 2 + z^{-1})$. Applying Algorithm 1, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = \frac{\sqrt{6}}{1080} \left[\begin{array}{cccccc} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z - \frac{1}{z}) & t_{16}(z - \frac{1}{z}) \\ 0 & 0 & 180(1 + z) & 180\sqrt{2} & t_{25}(1 - z) & t_{26}(1 - z) \\ \hline 360 & -\frac{b_{12}(z)}{\sqrt{2}} & -\frac{b_{13}(z)}{\sqrt{2}} & 0 & \frac{t_{15}}{\sqrt{2}}(\frac{1}{z} - z) & \frac{t_{16}}{\sqrt{2}}(\frac{1}{z} - z) \\ 0 & 0 & 90\sqrt{2}(1 + z) & -360 & \frac{t_{25}}{\sqrt{2}}(1 - z) & \frac{t_{26}}{\sqrt{2}}(1 - z) \\ \hline 0 & \sqrt{6}t_{13}(1 - z) & t_{53}(1 - z) & 0 & t_{55}(1 + z) & t_{56}(1 + z) \\ 0 & \frac{\sqrt{6}t_{12}}{2}(\frac{1}{z} - z) & \frac{\sqrt{6}t_{13}}{2}(\frac{1}{z} - z) & 0 & b_{65}(z) & b_{66}(z) \end{array} \right],$$

where $b_{65}(z) = -\sqrt{6}(5t_{15}(z + z^{-1}) + 3c_{12})/10$ and $b_{66}(z) = -\sqrt{6}t_{16}(z + z^{-1})/2 + c_{66}$. Note that $\mathcal{S}\mathsf{P}_e = [1, z, 1, z, -z, -1]^T [1, 1, 1, z^{-1}, -1, -1]$ and the coefficient support of P_e satisfies $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0 \leqslant m, \gamma \leqslant 2}$, we derive two high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ as follows:

$$\mathsf{a}_1(z) = \frac{\sqrt{2}}{1080} \begin{bmatrix} a_{11}^1(z) + a_{11}^1(z^{-1}) & a_{12}^1(z) + z^{-1}a_{12}^1(z^{-1}) \\ a_{21}^1(z) + z^3 a_{21}^1(z^{-1}) & a_{22}^1(z) + z^2 a_{22}^1(z^{-1}) \end{bmatrix},$$

$$\mathsf{a}_2(z) = \frac{\sqrt{6}}{1080} \begin{bmatrix} a_{11}^2(z) - z^3 a_{11}^2(z^{-1}) & a_{12}^2(z) - z^2 a_{12}^2(z^{-1}) \\ a_{21}^2(z) - a_{21}^2(z^{-1}) & a_{22}^2(z) - z^{-1}a_{22}^2(z^{-1}) \end{bmatrix},$$

where

$$a_{11}^1(z) = (47-7\sqrt{41})z^4 + 2(4+\sqrt{41})z^2 + 5(\sqrt{41}-11)z + 180;$$
$$a_{12}^1(z) = 2(2\sqrt{41}-17)z^3 + (\sqrt{41}-1)z^2 - 5(29+\sqrt{41});$$

$$a_{21}^1(z) = 3(37+3\sqrt{41})z + 3(23-3\sqrt{41})z^{-1};$$
$$a_{22}^1(z) = -180z + 3(21-\sqrt{41}) - 3(21-\sqrt{41})z^{-1};$$
$$a_{11}^2(z) = (43+17\sqrt{41})z + (67-7\sqrt{41})z^{-1};$$
$$a_{12}^2(z) = 11\sqrt{41} - 31 - (79+\sqrt{41})z^{-1};$$
$$a_{21}^2(z) = (47-7\sqrt{41})z^4 + 2(4+\sqrt{41})z^2 - 3(29+\sqrt{41})z;$$
$$a_{22}^2(z) = 2(2\sqrt{41}-17)z^3 + (\sqrt{41}-1)z^2 + 3(3+7\sqrt{41}).$$

Then the high-pass filters a_1, a_2 satisfy (3.13) with $c_1^1 = 0, c_2^1 = 1$, $\varepsilon_1^1 = \varepsilon_2^1 = 1$ and $c_1^2 = 1, c_2^2 = 0$, $\varepsilon_1^2 = \varepsilon_2^2 = -1$.

As demonstrated by the following example, our Algorithm 2 also applies to low-pass filters with symmetry patterns other than those in (3.3).

Example 3. Let $\mathsf{d} = 3$ and $r = 2$. A 3-band orthogonal low-pass filter a_0 with multiplicity 2 in [8] is given by

$$\mathsf{a}_0(z) = \frac{1}{702}\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix},$$

where

$$a_{11}(z) = (11-14\sqrt{17})z^2 + (29+8\sqrt{17})z + 234 + (85-16\sqrt{17})z^{-1} - (17+2\sqrt{17})z^{-2};$$
$$a_{12}(z) = (5\sqrt{17}-16)z^3 + (2+\sqrt{17})z^2 + 238 - 11\sqrt{17} + (136+29\sqrt{17})z^{-1};$$
$$a_{21}(z) = (136+29\sqrt{17})z^2 + (238-11\sqrt{17})z + (2+\sqrt{17})z^{-1} + (5\sqrt{17}-16)z^{-2};$$
$$a_{22}(z) = (-17-2\sqrt{17})z^3 + (85-16\sqrt{17})z^2 + 234z + 29 + 8\sqrt{17} + (11-14\sqrt{17})z^{-1}.$$

This low-pass filter a_0 does not satisfy (3.3). However, we can employ a very simple orthogonal transform $E := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ to a_0 so that the symmetry in (3.3) holds. That is, for $\widetilde{\mathsf{a}}_0(z) := E\mathsf{a}_0(z)E$, it is easy to verify that $\widetilde{\mathsf{a}}_0$ satisfies (3.3) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Construct $\mathsf{P}_{\widetilde{\mathsf{a}}_0} := [\widetilde{\mathsf{a}}_{0;0}, \widetilde{\mathsf{a}}_{0;1}, \widetilde{\mathsf{a}}_{0;2}]$ from $\widetilde{\mathsf{a}}_0$. The matrix U constructed by Lemma 1 from $\mathsf{P}_{\widetilde{\mathsf{a}}_0}$ is given by:

$$\mathsf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & z^{-1} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

Then $\mathsf{P} := \mathsf{P}_{\widetilde{\mathsf{a}}_0}\mathsf{U}$ satisfies $\mathcal{S}\mathsf{P} = [z^{-1}, -z^{-1}]^T[1, -1, -1, 1, 1, -1]$ and is given by

$$\mathsf{P} = c\begin{bmatrix} 234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \end{bmatrix},$$

where $c = \frac{\sqrt{6}}{1404}$ and t_{jk}'s are constants defined as follows:

$$t_{12} = 3(11 - \sqrt{17}); \quad t_{13} = 3(\sqrt{17} - 89); \quad t_{16} = 15\sqrt{2}(2 + \sqrt{17});$$
$$t_{21} = 13(\sqrt{17} - 17); \quad t_{22} = 6(2 + \sqrt{17}); \quad t_{23} = 6(37 - \sqrt{17});$$
$$t_{24} = -13(1 + \sqrt{17}); \quad t_{25} = -13\sqrt{2}(8 + \sqrt{17}); \quad t_{26} = -3\sqrt{2}(7 + 10\sqrt{17}).$$

Applying Algorithm 1 to P, we obtain a desired paraunitary matrix P_e as follows:

$$\mathsf{P}_e = c \left[\begin{array}{cccccc} 234(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & 0 & 117\sqrt{2}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \\ \hline t_{31}(1-\frac{1}{z}) & t_{32}(1+\frac{1}{z}) & t_{33}(1+\frac{1}{z}) & t_{34}(1-\frac{1}{z}) & t_{35}(1-\frac{1}{z}) & t_{36}(1+\frac{1}{z}) \\ t_{41}(1+\frac{1}{z}) & t_{42}(1-\frac{1}{z}) & t_{43}(1-\frac{1}{z}) & t_{44}(1+\frac{1}{z}) & -\sqrt{2}t_{41}(1+\frac{1}{z}) & t_{46}(1-\frac{1}{z}) \\ \hline \frac{2}{\sqrt{3}}t_{44} & 0 & 0 & -2\sqrt{3}t_{41} & -\frac{4}{\sqrt{6}}t_{44} & 0 \\ 0 & t_{62} & t_{63} & 0 & 0 & t_{66} \end{array}\right],$$

where all t_{jk}'s are constants given by:

$$t_{31} = -\sqrt{26}(61 + 25\sqrt{17})/4; \quad t_{32} = -3\sqrt{26}(397 + 23\sqrt{17})/52;$$
$$t_{33} = 3\sqrt{26}(553 + 23\sqrt{17})/52; \quad t_{34} = 25\sqrt{26}(1 + \sqrt{17})/4;$$
$$t_{35} = \sqrt{13}(25\sqrt{17} - 43)/2; \quad t_{36} = 15\sqrt{13}(23\sqrt{17} - 19)/26$$
$$t_{41} = 9\sqrt{26}(1 - 3\sqrt{17})/4; \quad t_{42} = -3\sqrt{26}(383 + 29\sqrt{17})/52;$$
$$t_{43} = 3\sqrt{26}(29\sqrt{17} + 227)/52; \quad t_{44} = 27\sqrt{26}(1 + \sqrt{17})/4;$$
$$t_{46} = 3\sqrt{13}(145\sqrt{17} - 61)/26; \quad t_{62} = 9\sqrt{78}(41\sqrt{17} - 9)/26;$$
$$t_{63} = 9\sqrt{78}(11\sqrt{17} + 9)/26; \quad t_{66} = 27\sqrt{3}(\sqrt{17} + 15)/\sqrt{13}.$$

Note that P_e satisfies $\mathcal{S}\mathsf{P}_e = [z^{-1}, -z^{-1}, -z^{-1}, z^{-1}, 1, -1]^T[1, -1, -1, 1, 1, -1]$ and we have $\operatorname{coeffsupp}([\mathsf{P}_e]_{:,j}) \subseteq \operatorname{coeffsupp}([\mathsf{P}]_{:,j})$ for all $1 \leqslant j \leqslant 6$. From the polyphase matrix $\mathcal{P} := \mathsf{P}_e\mathsf{U}^*$, we derive two high-pass filters $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ as follows:

$$\widetilde{\mathsf{a}}_1(z) = \frac{\sqrt{26}}{36504}\begin{bmatrix} a^1_{11}(z) - za^1_{11}(z^{-1}) & a^1_{12}(z) + za^1_{12}(z^{-1}) \\ a^1_{21}(z) + za^1_{21}(z^{-1}) & a^1_{22}(z) - za^1_{22}(z^{-1}) \end{bmatrix},$$
$$\widetilde{\mathsf{a}}_2(z) = \frac{\sqrt{78}}{4056}\begin{bmatrix} a^2_{11}(z) & a^2_{12}(z) \\ a^2_{21}(z) & a^2_{22}(z) \end{bmatrix},$$

where

$$a^1_{11}(z) = (433 - 128\sqrt{17})z^3 + 13(25\sqrt{17} - 43)z^2 - (1226 + 197\sqrt{17})z;$$
$$a^1_{12}(z) = (128\sqrt{17} - 433)z^3 + 15(23\sqrt{17} - 19)z^2 - (758 + 197\sqrt{17})z;$$
$$a^1_{21}(z) = 3(133 - 44\sqrt{17})z^3 + 117(3\sqrt{17} - 1)z^2 - 3(73\sqrt{17} + 94)z;$$
$$a^1_{22}(z) = 3(44\sqrt{17} - 133)z^3 + 3(145\sqrt{17} - 61)z^2 - 3(250 + 73\sqrt{17})z;$$
$$a^2_{11}(z) = 13(1 + \sqrt{17})(z^3 - 2z^2 + z);$$
$$a^2_{12}(z) = 13(3\sqrt{17} - 1)(z^3 - z);$$
$$a^2_{21}(z) = (9 + 11\sqrt{17})(z^3 - z);$$
$$a^2_{22}(z) = (41\sqrt{17} - 9)(z^3 + 24z^2/137 + 18\sqrt{17}z^2/137 + z).$$

Then the high-pass filters $\widetilde{\mathsf{a}}_1$ and $\widetilde{\mathsf{a}}_2$ satisfy (3.13) with $c_1^1 = c_2^1 = 1/2$, $\varepsilon_1^1 = -1, \varepsilon_2^1 = 1$ and $c_1^2 = c_2^2 = 3/2$, $\varepsilon_1^1 = 1, \varepsilon_2^1 = -1$, respectively.

Let $\mathsf{a}_1, \mathsf{a}_2$ be two high-pass filters constructed from $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ by $\mathsf{a}_1(z) := E\widetilde{\mathsf{a}}_1(z)E$ and $\mathsf{a}_2(z) := E\widetilde{\mathsf{a}}_2(z)E$. Then due to the orthogonality of E, $\{\mathsf{a}_0, \mathsf{a}_1, \mathsf{a}_2\}$ still forms a d-band filter bank with the perfect reconstruction property but their symmetry patterns are different to those of $\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$.

4. Proofs of Theorems 1 and 2. In this section, we shall prove Theorems 1 and 2. The key ingredient is to prove that the coefficient supports of $\mathsf{A}_1, \ldots, \mathsf{A}_J$ constructed in Algorithm 1 are all contained inside $[-1, 1]$. Note that each A_j takes the form $\mathsf{A}_j = (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}\mathsf{B}_{\mathsf{Q}_1}$. We first show that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1, 1]$ and then show that the coefficient support of $\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}$ is also contained inside $[-1, 1]$.

Let us first present a detailed construction for the unitary matrices $U_{\mathtt{f}}$ and U_G that are used in Algorithm 1. For a $1 \times n$ row vector $\mathtt{f}$ in $\mathbb{F}$ such that $\|\mathtt{f}\| \neq 0$, we define $n_{\mathtt{f}}$ to be the number of nonzero entries in $\mathtt{f}$ and $\mathbf{e}_j := [0, \ldots, 0, 1, 0, \ldots, 0]$ to be the jth unit coordinate row vector in $\mathbb{R}^n$. Let $E_{\mathtt{f}}$ be a permutation matrix such that $\mathtt{f}E_{\mathtt{f}} = [f_1, \ldots, f_{n_{\mathtt{f}}}, 0, \ldots, 0]$ with $f_j \neq 0$ for $j = 1, \ldots, n_{\mathtt{f}}$. We define

$$V_{\mathtt{f}} := \begin{cases} I_n, & \text{if } n_{\mathtt{f}} = 1; \\ \frac{\bar{f}_1}{|f_1|}\left(I_n - \frac{2}{\|v_{\mathtt{f}}\|^2} v_{\mathtt{f}}^* v_{\mathtt{f}}\right), & \text{if } n_{\mathtt{f}} > 1, \end{cases} \tag{4.1}$$

where $v_{\mathtt{f}} := \mathtt{f} - \frac{f_1}{|f_1|}\|\mathtt{f}\|\mathbf{e}_1$. Observing that $\|v_{\mathtt{f}}\|^2 = 2\|\mathtt{f}\|(\|\mathtt{f}\| - |f_1|)$, we can verify that $V_{\mathtt{f}}V_{\mathtt{f}}^* = I_n$ and $\mathtt{f}E_{\mathtt{f}}V_{\mathtt{f}} = \|\mathtt{f}\|\mathbf{e}_1$. Let $U_{\mathtt{f}} := E_{\mathtt{f}}V_{\mathtt{f}}$. Then $U_{\mathtt{f}}$ is unitary and satisfies $U_{\mathtt{f}} = [\frac{\mathtt{f}^*}{\|\mathtt{f}\|}, F^*]$ for some $(n-1) \times n$ matrix F in $\mathbb{F}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \ldots, 0]$. We also define $U_{\mathtt{f}} := I_n$ if $\mathtt{f} = \mathbf{0}$ and $U_{\mathtt{f}} := \emptyset$ if $\mathtt{f} = \emptyset$. Here, $U_{\mathtt{f}}$ plays the role of reducing the number of nonzero entries in $\mathtt{f}$. More generally, for an $r \times n$ nonzero matrix G of rank m in $\mathbb{F}$, employing the above procedure to each row of G, we can obtain an $n \times n$ unitary matrix U_G such that $GU_G = [R, \mathbf{0}]$ for some $r \times m$ lower triangular matrix R of rank m. If $G_1G_1^* = G_2G_2^*$, then the above procedure produces two matrices U_{G_1}, U_{G_2} such that $G_1U_{G_1} = [R, \mathbf{0}]$ and $G_2U_{G_2} = [R, \mathbf{0}]$ for some lower triangular matrix R of full rank. It is important to notice that the constructions of $U_{\mathtt{f}}$ and U_G only involve the nonzero entries of $\mathtt{f}$ and nonzero columns of G, respectively. In other words, we have

$$\begin{aligned} &[U_{\mathtt{f}}]_{j,:} = ([U_{\mathtt{f}}]_{:,j})^T = \mathbf{e}_j, && \text{if } [\mathtt{f}]_j = 0, \\ &[U_G]_{j,:} = ([U_G]_{:,j})^T = \mathbf{e}_j, && \text{if } [G]_{:,j} = \mathbf{0}. \end{aligned} \tag{4.2}$$

Next, we establish the following lemma, which is needed later to show that the coefficient support of $(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1, 1]$.

LEMMA 1. *Suppose* B *is an* $s \times s$ *paraunitary matrix such that* $\operatorname{coeffsupp}(\mathsf{B}) \subseteq [-1, 1]$ *and* $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^*\mathcal{S}\theta$ *with* $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ *for some nonnegative integers* $s_1, \ldots, s_4$ *such that* $s_1 + s_2 + s_3 + s_4 = s$. *Then the following statements hold.*

(1) *Let* p *be a* $1 \times s$ *row vector of Laurent polynomials with symmetry such that* $\mathsf{p}\mathsf{p}^* = 1$, $\operatorname{coeffsupp}(\mathsf{p}) = [k_1, k_2]$ *with* $k_2 - k_1 \geqslant 2$, *and* $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$ *for some* $\varepsilon \in \{-1, 1\}$ *and* $c \in \{0, 1\}$. *Let* $\mathsf{q} := \mathsf{p}\mathsf{B}$. *If* $\operatorname{coeffsupp}(\mathsf{q}) = \operatorname{coeffsupp}(\mathsf{p})$, *then* $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{q}}) \subseteq [-1, 1]$, *where* B_{q} *is constructed with respect to* q *as in section 2.*

(2) *Let* $\mathsf{p}_1, \mathsf{p}_2$ *be two* $1 \times s$ *row vectors of Laurent polynomials with symmetry such that* $\mathsf{p}_{j_1}\mathsf{p}_{j_2}^* = \delta(j_1 - j_2)$ *for* $j_1, j_2 = 1, 2$, $\mathcal{S}\mathsf{p}_1 = \varepsilon_1\mathcal{S}\theta$ *and* $\mathcal{S}\mathsf{p}_2 = \varepsilon_2 z\mathcal{S}\theta$

for some $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$, *and* $\operatorname{coeffsupp}(\mathsf{p}_1) = \operatorname{coeffsupp}(\mathsf{p}_2) \subseteq [-k, k]$ *with* $k \geqslant 1$. *Let* $\mathsf{q}_1 := \mathsf{p}_1\mathsf{B}$ *and* $\mathsf{q}_2 := \mathsf{p}_2\mathsf{B}$. *If* $\operatorname{coeffsupp}(\mathsf{q}_1) = [-k, k-1]$ *and* $\operatorname{coeffsupp}(\mathsf{q}_2) = [-k+1, k]$, *then* $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1, 1]$, *where* $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ *is constructed with respect to the pair* $(\mathsf{q}_1, \mathsf{q}_2)$ *as in section 2.*

Proof. Due to $\mathcal{S}\mathsf{p} = \varepsilon z^c \mathcal{S}\theta$, as we discussed in section 2, there is an $\mathsf{U}_{\mathsf{p},\varepsilon}$ such that $\mathsf{p}\mathsf{U}_{\mathsf{p},\varepsilon}$ takes the form in (2.3). Since $\mathsf{U}_{\mathsf{p},\varepsilon}$ is a product of a permutation matrix and a diagonal matrix of monomials, we shall consider the case that $\mathsf{U}_{\mathsf{p},\varepsilon} = I_s$, while the proofs for other cases of $\mathsf{U}_{\mathsf{p},\varepsilon}$ can be obtained accordingly. Then p takes the standard form in (2.3) with $\mathtt{f}_1 \neq \mathbf{0}$. In this case, $s_1 > 0$ and $s_2 > 0$ due to $\|\mathtt{f}_1\| = \|\mathtt{f}_2\| \neq 0$. By our assumptions, $\mathsf{q} := \mathsf{p}\mathsf{B}$ must take the following form:

$$\begin{aligned}\mathsf{q} := \mathsf{p}\mathsf{B} =& [\widetilde{\mathtt{f}}_1, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2] z^{k_1} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4] z^{k_1+1} + \sum_{n=k_1+2}^{k_2-2} \operatorname{coeff}(\mathsf{p}\mathsf{B}, n) z^n \\ &+ [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2] z^{k_2-1} + [\widetilde{\mathtt{f}}_1, \widetilde{\mathtt{f}}_2, \mathbf{0}, \mathbf{0}] z^{k_2}\end{aligned}$$

with $\widetilde{\mathtt{f}}_1 \neq \mathbf{0}$. Then B_{q} is given by (2.5) with $\mathtt{f}_1$, $\mathtt{f}_2$, $\mathtt{g}_1$, $\mathtt{g}_2$, F_1, F_2, G_1, G_2 being replaced by $\widetilde{\mathtt{f}}_1$, $\widetilde{\mathtt{f}}_2$, $\widetilde{\mathtt{g}}_1$, $\widetilde{\mathtt{g}}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{G}_1$, $\widetilde{G}_2$ respectively and all constants $c_{\widetilde{\mathtt{f}}_1}, c_{\widetilde{\mathtt{g}}_1}, c_{\widetilde{\mathtt{g}}_2}, c_0, c, c_{\widetilde{\mathtt{g}}_1'}, c_{\widetilde{\mathtt{g}}_2'}$ being defined accordingly.

Also, due to the symmetry pattern and $\operatorname{coeffsupp}(\mathsf{B}) \subseteq [-1, 1]$, B is of the form:

$$\mathsf{B} = \left[\begin{array}{c|c|c|c} A_1(z+\frac{1}{z}) + D_1 & A_3(z-\frac{1}{z}) & B_3(1+\frac{1}{z}) & B_4(1-\frac{1}{z}) \\ \hline A_2(z-\frac{1}{z}) & A_4(z+\frac{1}{z}) + D_2 & C_3(1-\frac{1}{z}) & C_4(1+\frac{1}{z}) \\ \hline B_1(1+z) & C_1(1-z) & A_5(z+\frac{1}{z}) + D_3 & A_7(z-\frac{1}{z}) \\ \hline B_2(1-z) & C_2(1+z) & A_6(z-\frac{1}{z}) & A_8(z+\frac{1}{z}) + D_4 \end{array}\right], \tag{4.3}$$

where A_j's, B_j's, C_j's and D_j's are all constant matrices in $\mathbb{F}$ and D_j is of size $s_j \times s_j$ for $j = 1, \ldots, 4$.

Let $\mathcal{I} := \{1, s_1+1, (1-\delta(s_3))(s_1+s_2+1), (1-\delta(s_4))(s_1+s_2+s_3+1)\}$ be an index set. It is easy to verify that $\operatorname{coeffsupp}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}) \subseteq [-1, 1]$ for all $j \notin \mathcal{I}$. Hence, by $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{q}}) \subseteq [-2, 2]$, we only need to compute $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, 2)$ and $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, -2)$ for those $j \in \mathcal{I}$. Let us show that $\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,j}, 2) = \mathbf{0}$ for $j = 1$, i.e., the coefficient vector of z^2 for the first column of $\mathsf{B}\mathsf{B}_{\mathsf{q}}$ is $\mathbf{0}$. By $\operatorname{coeff}(\mathsf{p}\mathsf{B}, k_1) = \operatorname{coeff}(\mathsf{p}, k_1+1)\operatorname{coeff}(\mathsf{B}, -1) + \operatorname{coeff}(\mathsf{p}, k_1)\operatorname{coeff}(\mathsf{B}, 0)$, we have

$$\begin{aligned}
\widetilde{\mathtt{f}}_1 &= \mathtt{f}_3 A_1 + \mathtt{f}_4 A_2 + \mathtt{f}_1 D_1 + \mathtt{g}_1 B_1 - \mathtt{g}_2 B_2; \\
\widetilde{\mathtt{f}}_2 &= \mathtt{f}_3 A_3 + \mathtt{f}_4 A_4 + \mathtt{f}_2 D_2 - \mathtt{g}_1 C_1 + \mathtt{g}_2 C_2; \\
\widetilde{\mathtt{g}}_1 &= \mathtt{f}_3 B_3 + \mathtt{f}_4 C_3 + \mathtt{g}_3 A_5 + \mathtt{g}_4 A_6 + \mathtt{f}_1 B_3 - \mathtt{f}_2 C_3 + \mathtt{g}_1 D_3; \\
\widetilde{\mathtt{g}}_2 &= \mathtt{f}_3 B_4 + \mathtt{f}_4 C_4 + \mathtt{g}_3 A_7 + \mathtt{g}_4 A_8 - \mathtt{f}_1 B_4 + \mathtt{f}_2 C_4 + \mathtt{g}_2 D_4.
\end{aligned} \tag{4.4}$$

Similarly, by $\operatorname{coeff}(\mathsf{B}\mathsf{B}_{\mathsf{q}}, 2) = \operatorname{coeff}(\mathsf{B}, 1)\operatorname{coeff}(\mathsf{B}_{\mathsf{q}}, 1)$, we have

$$\operatorname{coeff}([\mathsf{B}\mathsf{B}_{\mathsf{q}}]_{:,1}, 2) = \frac{1}{c}\begin{bmatrix} A_1 & A_3 & \mathbf{0} & \mathbf{0} \\ A_2 & A_4 & \mathbf{0} & \mathbf{0} \\ B_1 & -C_1 & A_5 & A_7 \\ -B_2 & C_2 & A_6 & A_8 \end{bmatrix} \begin{bmatrix} \widetilde{\mathtt{f}}_1^* \\ -\widetilde{\mathtt{f}}_2^* \\ \widetilde{\mathtt{g}}_1^* \\ -\widetilde{\mathtt{g}}_2^* \end{bmatrix} = \frac{1}{c}\begin{bmatrix} A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^* \\ A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^* \\ B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^* \\ -B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* \end{bmatrix}.$$

Due to $\mathsf{B}\mathsf{B}^* = I_s$, we obtain

$$\begin{cases} A_1A_1^* - A_3A_3^* = \mathbf{0},\ A_1A_2^* - A_3A_4^* = \mathbf{0}; \\ A_1D_1^* + D_1A_1^* + B_3B_3^* - B_4B_4^* = \mathbf{0}; \\ D_1A_2^* - A_3D_2^* + B_3C_3^* - B_4C_4^* = \mathbf{0}; \\ A_1B_1^* + A_3C_1^* + B_3A_5^* - B_4A_7^* = \mathbf{0}; \\ -A_1B_2^* - A_3C_2^* + B_3A_6^* - B_4A_8^* = \mathbf{0}. \end{cases}$$

Applying the above identities to $A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^*$ and using (4.4), we get

$$\begin{aligned}
A_1\widetilde{\mathtt{f}}_1^* - A_3\widetilde{\mathtt{f}}_2^* &= A_1(\mathtt{f}_3A_1 + \mathtt{f}_4A_2 + \mathtt{f}_1D_1 + \mathtt{g}_1B_1 - \mathtt{g}_2B_2)^* \\
&\quad - A_3(\mathtt{f}_3A_3 + \mathtt{f}_4A_4 + \mathtt{f}_2D_2 - \mathtt{g}_1C_1 + \mathtt{g}_2C_2)^* \\
&= (A_1A_1^* - A_3A_3^*)\mathtt{f}_3^* + (A_1A_2^* - A_3A_4^*)\mathtt{f}_4^* + (A_1D_1^*)\mathtt{f}_1^* \\
&\quad + (-A_3D_2^*)\mathtt{f}_2^* + (A_1B_1^* + A_3C_1^*)\mathtt{g}_1^* - (A_1B_2^* + A_3C_2^*)\mathtt{g}_2^* \\
&= -(D_1A_1^* + B_3B_3^* - B_4B_4^*)\mathtt{f}_1^* - (D_1A_2^* + B_3C_3^* - B_4C_4^*)\mathtt{f}_2^* \\
&\quad - (B_3A_5^* - B_4A_7^*)\mathtt{g}_1^* - (B_3A_6^* - B_4A_8^*)\mathtt{g}_2^* \\
&= -D_1(\mathtt{f}_1A_1 + \mathtt{f}_2A_2)^* - B_3(\mathtt{f}_1B_3 + \mathtt{f}_2C_3 + \mathtt{g}_1A_5 + \mathtt{g}_2A_6)^* \\
&\quad + B_4(\mathtt{f}_1B_4 + \mathtt{f}_2C_4 + \mathtt{g}_1A_7 + \mathtt{g}_2A_8)^* = \mathbf{0},
\end{aligned}$$

where the last above identity follows by $\mathrm{coeff}(\mathsf{pB}, k_2+1) = \mathrm{coeff}(\mathsf{pB}, k_1-1) = \mathbf{0}$. Similarly, we can show that $A_2\widetilde{\mathtt{f}}_1^* - A_4\widetilde{\mathtt{f}}_2^* = \mathbf{0}$, $B_1\widetilde{\mathtt{f}}_1^* + C_1\widetilde{\mathtt{f}}_2^* + A_5\widetilde{\mathtt{g}}_1^* - A_7\widetilde{\mathtt{g}}_2^* = \mathbf{0}$, and $-B_2\widetilde{\mathtt{f}}_1^* - C_1\widetilde{\mathtt{f}}_2^* + A_6\widetilde{\mathtt{g}}_1^* - A_8\widetilde{\mathtt{g}}_2^* = \mathbf{0}$. Hence, $\mathrm{coeff}([\mathsf{BB}_{\mathsf{q}}]_{:,1}, 2) = \mathbf{0}$. By similar computations as above and using the paraunitary property of B, we have $\mathrm{coeff}([\mathsf{BB}_{\mathsf{q}}]_{:,j}, \pm 2) = \mathbf{0}$ for all $j \in \mathcal{I}$. Therefore, we conclude that $\mathrm{coeffsupp}(\mathsf{BB}_{\mathsf{q}}) \subseteq [-1,1]$. Item (1) holds.

For item (2), up to a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ as in section 2, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ takes the form in (2.10). Since B takes the form in (4.3), to show that the coefficient support of $\mathsf{BB}_{(-k,k)}$ is contained inside $[-1,1]$, we need to show that all the coefficient vectors $A_1\widetilde{\mathtt{g}}_1^* - A_3\widetilde{\mathtt{g}}_2^*$, $A_2\widetilde{\mathtt{g}}_1^* - A_4\widetilde{\mathtt{g}}_2^*$, $A_5\widetilde{\mathtt{g}}_3^* - A_7\widetilde{\mathtt{g}}_4^*$, and $A_6\widetilde{\mathtt{g}}_3^* - A_8\widetilde{\mathtt{g}}_4^*$ are zero. Again, using the paraunitary property of B and expressing $\widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2, \widetilde{\mathtt{g}}_3, \widetilde{\mathtt{g}}_4$ in terms of the original vectors from $\mathsf{p}_1, \mathsf{p}_2$ similar to (4.4), we conclude that $\mathrm{coeffsupp}(\mathsf{BB}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq [-1,1]$. $\square$

With the result of Lemma 1, the next lemma shows that the coefficient support of $\mathsf{B} := (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$. Moreover, the next lemma shows that the coefficient support of $\mathsf{A} := \mathsf{BB}_{\mathsf{Q}_1}$ is also contained inside $[-1,1]$.

Lemma 2. *Suppose* Q *is an* $r \times s$ *matrix of Laurent polynomials such that* $\mathsf{QQ}^* = I_r$*,* $\mathcal{S}\mathsf{Q}$ *satisfies* (2.1)*, and* $\mathrm{coeffsupp}(\mathsf{Q}) = [k_1, k_2]$ *with* $k_2 - k_1 \geqslant 1$*. Then there exists an* $s \times s$ *paraunitary matrix* A *of Laurent polynomials with symmetry such that*

(1) $\mathrm{coeffsupp}(\mathsf{A}) \subseteq [-1,1]$ *and* $|\mathrm{coeffsupp}(\mathsf{QA})| \leqslant |\mathrm{coeffsupp}(\mathsf{Q})| - |\mathrm{coeffsupp}(\mathsf{A})|$*;*

(2) *if the* j*th column* $\mathsf{p} := [\mathsf{Q}]_{:,j}$ *of* Q *satisfies* $\mathrm{coeff}(\mathsf{p}, k_1) = \mathrm{coeff}(\mathsf{p}, k_2) = \mathbf{0}$*, then* $[\mathsf{A}]_{j,:} = ([\mathsf{A}]_{:,j})^T = \mathbf{e}_j$*. That is, any entry in the* j*th row or* j*th column of* A *is zero except that the* (j,j)*-entry* $[\mathsf{A}]_{j,j} = 1$*;*

(3) $\mathcal{S}\mathsf{A} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^T[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ *for some nonnegative integers* $s_1', \ldots, s_4'$ *such that* $s_1' + s_2' + s_3' + s_4' = s$*.*

Proof. Let $\mathsf{A} = (\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}\mathsf{B}_{\mathsf{Q}_1}$ be constructed as in Algorithm 1, where $\mathsf{Q}_1 := \mathsf{Q}(\mathsf{B}_1 \cdots \mathsf{B}_r)\mathsf{B}_{(-k,k)}$, $\mathsf{B}_{(-k,k)}$ is constructed in the inner `while` loop of Algorithm 1, and $\mathsf{B}_1, \ldots, \mathsf{B}_r$ is constructed in the `for` loop of Algorithm 1. If $k_2 \neq -k_1$, then $\mathsf{B}_1 = \cdots = \mathsf{B}_r = \mathsf{B}_{(-k,k)} = I_s$ and A is simply $\mathsf{B}_{\mathsf{Q}_1}$, where $\mathsf{Q}_1 = \mathsf{Q}$ is of the form in (2.8) with either $\mathrm{coeff}(\mathsf{Q}_1, -k) = \mathbf{0}$ or $\mathrm{coeff}(\mathsf{Q}_1, k) = \mathbf{0}$. In this case, by the construction of $\mathsf{B}_{\mathsf{Q}_1}$ as in section 2, all items in Lemma 2 hold. We are already done. So, without loss of generality, we assume that $k_2 = -k_1 = k$.

We first show that the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$ is contained inside $[-1,1]$. Let $\mathsf{p}_j := [\mathsf{Q}]_{j,:}$, $\mathsf{B}_0 := I_s$, and $\mathsf{q}_j := \mathsf{p}_j\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ for $j = 1, \ldots, r$. Suppose we already show that $\mathrm{coeffsupp}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) \subseteq [-1,1]$ for $j \geqslant 1$. Then, according to Algorithm 1, $\mathsf{B}_j = \mathsf{B}_{\mathsf{q}_j}$ if $\mathrm{coeffsupp}(\mathsf{p}_j) = \mathrm{coeffsupp}(\mathsf{q}_j)$, $|\mathrm{coeffsupp}(\mathsf{q}_j)| \geqslant 2$, and one of $\mathrm{coeff}(\mathsf{q}_j, k)$ and $\mathrm{coeff}(\mathsf{q}_j, -k)$ is nonzero; otherwise $\mathsf{B}_j = I_s$. Note that $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}$ is paraunitary and satisfies $\mathcal{S}(\mathsf{B}_0 \cdots \mathsf{B}_{j-1}) = (\mathcal{S}\theta)^*\mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. By item (1) of Lemma 1, the coefficient support of $\mathsf{B}_0 \cdots \mathsf{B}_{j-1}\mathsf{B}_j$ is also contained inside $[-1,1]$. By induction, the coefficient support of $\mathsf{B}_1 \cdots \mathsf{B}_r$ is contained inside

$[-1,1]$. Moreover, $\mathsf{B}_1\cdots\mathsf{B}_r$ takes the form in (4.3). Next, since $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1,\mathsf{q}_2)$ of $\mathsf{Q}_0 := \mathsf{Q}(\mathsf{B}_1\cdots\mathsf{B}_r)$, by applying induction again and using item (2) of Lemma 1, we conclude that the coefficient support of $\mathsf{B} := (\mathsf{B}_1\cdots\mathsf{B}_r)\mathsf{B}_{(-k,k)}$ is contained inside $[-1,1]$.

Due to the Property (P1), (P2) of B_{q} and (P3), (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_1,\ldots,\mathsf{B}_r$ and $\mathsf{B}_{(-k,k)}$ reduce Q of the form in (2.7) to $\mathsf{Q}_1 = \mathsf{Q}(\mathsf{B}_1\cdots\mathsf{B}_r)\mathsf{B}_{(-k,k)} = \mathsf{Q}\mathsf{B}$ of the form in (2.8) with at least one of $\mathrm{coeff}(\mathsf{Q}_1,-k)$ and $\mathrm{coeff}(\mathsf{Q}_1,k)$ being $\mathbf{0}$. As constructed in section 2, $\mathsf{B}_{\mathsf{Q}_1} = I_s$ for the case that $\mathrm{coeff}(\mathsf{Q}_1,-k) = \mathrm{coeff}(\mathsf{Q}_1,k) = \mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} = \mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ for the case $\mathrm{coeff}(\mathsf{Q}_1,k)\neq\mathbf{0}$, or $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ for the case that $\mathrm{coeff}(\mathsf{Q}_1,-k)\neq\mathbf{0}$. We next show that $\mathrm{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1})\subseteq[-1,1]$.

Let Q take the form in (2.7) and Q_1 take the form in (2.8) with $\mathrm{coeff}(\mathsf{Q}_1,k)\neq\mathbf{0}$. Then $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with U_1, W_1, and E being constructed as in section 2. Note that B takes the form in (4.3). Define

$$[G_1,G_2,F_3,F_4,G_5,G_6,F_7,F_8] := \begin{bmatrix} G_{11} & G_{21} & F_{31} & F_{41} & G_{51} & G_{61} & F_{71} & F_{81} \\ G_{12} & G_{22} & F_{32} & F_{42} & G_{52} & G_{62} & F_{72} & F_{82} \end{bmatrix}.$$

By $\mathrm{coeff}(\mathsf{Q}_1,k) = \mathrm{coeff}(\mathsf{Q},k-1)\mathrm{coeff}(\mathsf{B},1) + \mathrm{coeff}(\mathsf{Q},k)\mathrm{coeff}(\mathsf{B},0)$, we have

$$\begin{aligned} \widetilde{G}_1 &= G_5A_1 + G_6A_2 + F_7B_1 - F_8B_2 + G_1D_1 + F_3B_1 + F_4B_2; \\ \widetilde{G}_2 &= G_5A_3 + G_6A_4 - F_7C_1 + F_8C_2 + G_2D_2 + F_3C_1 + F_4C_2; \\ \mathbf{0} &= F_7A_5 + F_8A_6 + G_1B_3 + G_2C_3 + F_3D_3 =: \widetilde{F}_3; \\ \mathbf{0} &= F_7A_7 + F_8A_8 + G_1B_4 + G_2C_4 + F_4D_4 =: \widetilde{F}_4, \end{aligned} \tag{4.5}$$

where $\widetilde{G}_1,\widetilde{G}_2$ are matrices defined in (2.11). Then $U_1 = \mathrm{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and W_1 is defined as in (2.12). By the coefficient supports of B and $\mathsf{B}_{\mathsf{Q}_1}$, we only need to check that $\mathrm{coeff}(\mathsf{B}\,\mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-2) = \mathbf{0}$. Let $V_{11},V_{12},V_{21},V_{22}$ be diagonal matrices of size $s_1\times s_1$, $s_1\times s_2$, $s_2\times s_1$, $s_2\times s_2$, respectively, and satisfy $\mathrm{diag}(V_{j\ell}) = [\mathbf{1}_{m_1},\mathbf{0}]$ for $j,\ell=1,2$, where m_1 is the rank of $\widetilde{G}_1$. Then

$$\begin{aligned} &\mathrm{coeff}(\mathsf{B}\,\mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-2) = \mathrm{coeff}(\mathsf{B},-1)\cdot\mathrm{coeff}(\mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4}),-1) \\ &= \begin{bmatrix} A_1 & -A_3 & B_3 & -B_4 \\ -A_2 & A_4 & -C_3 & C_4 \\ \mathbf{0} & \mathbf{0} & A_5 & -A_7 \\ \mathbf{0} & \mathbf{0} & -A_6 & A_8 \end{bmatrix} \begin{bmatrix} U_{\widetilde{G}_1}V_{11} & U_{\widetilde{G}_1}V_{12} & \mathbf{0} & \mathbf{0} \\ U_{\widetilde{G}_2}V_{21} & U_{\widetilde{G}_2}V_{22} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \end{aligned}$$

Thus, we need to show $A_1U_{\widetilde{G}_1}V_{1j} - A_3U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$ and $A_2U_{\widetilde{G}_1}V_{1j} - A_4U_{\widetilde{G}_2}V_{2j} = \mathbf{0}$, for $j=1,2$, which is equivalent to showing that $V_{j1}U^*_{\widetilde{G}_1}A_1^* - V_{j2}U^*_{\widetilde{G}_2}A_3^* = \mathbf{0}$ and $V_{j1}U^*_{\widetilde{G}_1}A_2^* - V_{j2}U^*_{\widetilde{G}_2}A_4^* = \mathbf{0}$ for $j=1,2$. Since $\widetilde{G}_1U_{\widetilde{G}_1} = [R,\mathbf{0}]$ and $\widetilde{G}_2U_{\widetilde{G}_2} = [R,\mathbf{0}]$, for some lower triangular matrix R of full rank m_1, it is equivalent to proving that

$\widetilde{G}_1 A_1^* - \widetilde{G}_2 A_3^* = \mathbf{0}$ and $\widetilde{G}_1 A_2^* - \widetilde{G}_2 A_4^* = \mathbf{0}$. By (4.5), we have,

$$\begin{aligned}
\widetilde{G}_1 A_1^* - \widetilde{G}_2 A_3^* &= \widetilde{G}_1 A_1^* - \widetilde{G}_2 A_3^* + \widetilde{F}_3 B_3^* - \widetilde{F}_4 B_4^* \\
&= (G_5 A_1 + G_6 A_2 + F_7 B_1 - F_8 B_2 + G_1 D_1 + F_3 B_1 + F_4 B_2) A_1^* \\
&\quad - (G_5 A_3 + G_6 A_4 - F_7 C_1 + F_8 C_2 + G_2 D_2 + F_3 C_1 + F_4 C_2) A_3^* \\
&\quad + (F_7 A_5 + F_8 A_6 + G_1 B_3 + G_2 C_3 + F_3 D_3) B_3^* \\
&\quad - (F_7 A_7 + F_8 A_8 + G_1 B_4 + G_2 C_4 + F_4 D_4) B_4^* \\
&= G_5 (A_1 A_1^* - A_3 A_3^*) + G_6 (A_2 A_1^* - A_4 A_3^*) \\
&\quad + F_7 (B_1 A_1^* + C_1 A_3^* + A_5 B_3^* - A_7 B_4^*) \\
&\quad + F_8 (-B_2 A_1^* - C_2 A_3^* + A_6 B_3^* - A_8 B_4^*) \\
&\quad + G_1 (D_1 A_1^* + B_3 B_3^* - B_4 B_4^*) + G_2 (-D_2 A_3^* + C_3 B_3^* - C_4 B_4^*) \\
&\quad + F_3 (B_1 A_1^* - C_1 A_3^* + D_3 B_3^*) + F_4 (B_2 A_1^* - C_2 A_3^* - D_4 B_4^*) = \mathbf{0},
\end{aligned}$$

where the last identity follows from $\mathsf{B}\mathsf{B}^* = I_s$ and $\operatorname{coeff}(\mathsf{Q}\mathsf{B}, k+1) = \mathbf{0}$. Similarly, $\widetilde{G}_1 A_2^* - \widetilde{G}_2 A_4^* = \mathbf{0}$. The computation for showing $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$ with $\mathsf{B}_{\mathsf{Q}_1} = \operatorname{diag}(I_{s_1+s_2}, U_3 \mathsf{W}_3) E$ is similar. Consequently, $\operatorname{coeffsupp}(\mathsf{B}\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-1,1]$. Therefore, item (1) holds. Item (2) is due to the property (4.2) of $U_{\mathtt{f}}$ and U_G.

Note that $\mathcal{S}\mathsf{B} = (\mathcal{S}\theta)^* \mathcal{S}\theta$ with $\mathcal{S}\theta = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$. And by the construction of $\mathsf{B}_{\mathsf{Q}_1}$, $\mathcal{S}\mathsf{B}_{\mathsf{Q}_1} = (\mathcal{S}\theta)^*[\mathbf{1}_{s_1'}, -\mathbf{1}_{s_2'}, z^{-1}\mathbf{1}_{s_3'}, -z^{-1}\mathbf{1}_{s_4'}]$ for some nonnegative integers $s_1', \ldots, s_4'$ depending on the rank of $\widetilde{G}_1$ or $\widetilde{G}_3$ (see section 2). Consequently, item (3) holds. This also completes the proof of Algorithm 1. $\square$

Now, we are ready to prove Theorems 1 and 2.

Proof of Theorems 1 and 2: The sufficiency part of Theorem 2 is obvious. We only need to show the necessary part. Suppose $\mathcal{S}\mathsf{P} = (\mathcal{S}\theta_1)^* \mathcal{S}\theta_2$. Let $\mathsf{Q} := \mathsf{U}_{\mathcal{S}\theta_1}^* \mathsf{P} \mathsf{U}_{\mathcal{S}\theta_2}$ and $\operatorname{coeffsupp}(\mathsf{Q}) := [k_1, k_2]$. Then $\mathcal{S}\mathsf{Q}$ satisfies (2.1). By Lemma 2, the step of support reduction in Algorithm 1 produces a sequence of paraunitary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_J$ with coefficient support contained inside $[-1,1]$ such that $\mathsf{Q}\mathsf{A}_1 \cdots \mathsf{A}_J = [I_r, \mathbf{0}]$. Due to item (1) of Lemma 2, $J \leqslant \lceil \frac{k_2 - k_1}{2} \rceil$. Let $\mathsf{P}_j := \mathsf{A}_j^*$, $\mathsf{P}_0 := \mathsf{U}_{\mathcal{S}\theta_2}^*$ and $\mathsf{P}_{J+1} := \operatorname{diag}(\mathsf{U}_{\mathcal{S}\theta_1}, I_{s-r})$. Then $\mathsf{P}_e := \mathsf{P}_{J+1} \mathsf{P}_J \cdots \mathsf{P}_1 \mathsf{P}_0$ satisfies $[I_r, \mathbf{0}]\mathsf{P}_e = \mathsf{P}$. By item (3) of Lemma 2, $(\mathsf{P}_{j+1}, \mathsf{P}_j)$ has mutually compatible symmetry for all $0 \leqslant j \leqslant J$. The claim that $|\operatorname{coeffsupp}([\mathsf{P}_e]_{k,j})| \leqslant \max_{1 \leqslant n \leqslant r} |\operatorname{coeffsupp}([\mathsf{P}]_{n,j})|$ for $1 \leqslant j, k \leqslant s$ follows from item (2) of Lemma 2. Hence, all claims in Theorems 1 and 2 have been verified. $\square$

REFERENCES

[1] C. K. Chui and J. A. Lian, Construction of compactly supported symmetric and antisymmetric orthonormal wavelets with scale = 3, *Appl. Comput. Harmon. Anal.*, **2** (1995), 21–51.

[2] L. H. Cui, Some properties and construction of multiwavelets related to different symmetric centers, *Math. Comput. Simul.*, **70** (2005), 69–89.

[3] I. Daubechies, Ten Lectures on Wavelets, SIAM, CBMS Series, 1992.

[4] G. Donovan, J. Geronimo, and D. Hardin, Intertwining Multiresolution analysis and the construction of piecewise-polynomial wavelets, *SIAM J. Math. Anal.*, **27** (1996), 1791–1815.

[5] J. Geronimo, D. P. Hardin, and P. Massopust, Fractal functions and wavelet expansions based on several scaling functions, *J. Approx. Theory*, **78** (1994) 373–401.

[6] B. Han, Symmetric orthonormal scaling functions and wavelets with dilation factor 4, *Adv. Comput. Math.*, **8** (1998), 221–247.

[7] B. Han, Matrix extension with symmetry and applications to symmetric orthonormal complex M-wavelets, *J. Fourier Anal. Appl.*, to appear.

[8] B. Han, S. Kwon and X. S. Zhuang, Generalized interpolating refinable function vectors, *J. Comput. Appl. Math.*, **227** (2009), 254–270.

[9] B. Han and H. Ji, Compactly supported orthonormal complex wavelets with dilation 4 and symmetry, *Appl. Comput. Harmon. Anal.*, **26** (2009), 422–431.

[10] B. Han and Q. Mo, Splitting a matrix of Laurent polynomials with symmetry and its application to symmetric framelet filter banks. *SIAM J. Matrix Anal. Appl.*, **26** (2004), 97–124.

[11] B. Han and X. S. Zhuang, Analysis and construction of multivariate interpoalting refinable function vectors, *Acta Appl. Math.*, **107** (2009), 143–171.

[12] H. Ji and Z. Shen, Compactly supported (bi)orthogonal wavelets generated by interpolatory refinable functions, *Adv. Comput. Math.*, **11** (1999), 81–104.

[13] Q. T. Jiang, Parameterization of symmetric orthogonal multifilter banks with different filter lengths, *Linear Algebra Appl.*, **311** (2000) 79–96.

[14] Q. T. Jiang, Symmetric paraunitary matrix extension and parameterization of symmetric orthogonal multifilter banks, *SIAM J. Matrix Anal. Appl.*, **22** (2001) 138–166.

[15] W. Lawton, S. L. Lee, and Z. Shen, An algorithm for matrix extension and wavelet construction, *Math. Comp.*, **65** (1996), 723–737.

[16] A. Petukhov, Construction of symmetric orthogonal bases of wavelets and tight wavelet frames with integer dilation factor. *Appl. Comput. Harmon. Anal.*, **17** (2004), 198–210.

[17] Z. Shen, Refinable function vectors, *SIAM J. Math. Anal.*, **29** (1998), 235–250.

[18] R. Turcajova, An algorithm for the construction of symmetric orthogonal multiwavelets, *SIAM J. Matrix Anal. Appl.*, **25** (2003), 532–550.

[19] P. P. Vaidyanathan, Multirate Systems and Filter Banks, Prentice Hall, New Jersey, 1992.

[20] D. C. Youla and P. F. Pickel, The Quinen-Suslin theorem and the structure of n-dimesional elementary polynomial matrices, *IEEE Trans. Circ. Syst.*, **31** (1984), 513–518.